$\eta \longrightarrow 3\pi$

Diplomarbeit

der Philosophisch-naturwissenschaftlichen Fakultät

der Universität Bern

vorgelegt von

Manuel Walker

1998

Leiter der Arbeit:

Prof. Heinrich Leutwyler
Institut für theoretische Physik, Universität Bern

Menschliche Unwissenheit

Wie sehr der Mensch nach Wissenschaft verborgner Dinge ringt,
So bleibt ihm doch unzählig viel, davon er sagt: Mich dünkt.

Friedrich von Logau

Acknowledgments

First of all I am indebted to Professor Heinrich Leutwyler for his excellent support and advice during my diploma. To my parents I am very grateful for having financed my studies and for having invoked my interest in nature and science. I also have to thank my mother and sister for patiently listening to physical discussions between my father and me.

Contents

1 Introduction

If you want to find out your mass (many people don't), this seems to be a trivial matter: you stand on a scale and read off the result (which is believed to be systematically too high).
Whenever a new particle can be added to the particle zoo, one tries to find out as much as possible about its properties (as mass, charge, spin, decay rate and modes, etc.). If we want to measure the mass of an electron, for instance, this can only be done indirectly: From Milikan's experiment we get the elementary charge e and with the help of a beam tube one is able to measure the ratio m_e/e.
When determining the electron mass, one makes use of the crucial fact that (almost) free electrons do exist. If we are dealing with quark masses, we encounter a problem: due to confinement, free quarks simply don't exist. To get some knowledge concerning their masses, we are forced to go a much more indirect way than in the case of m_e. One has to gather information giving hints concerning quark masses or quark mass ratios from many sides in order to get reliable predictions.
Chiral perturbation theory (χPT), an effective field theory describing the low energy regime of QCD, does not allow to determine the individual quark masses phenomenologically. This is because the low energy constant B occurring in the effective Lagrangian (see section 2.4) can not be measured directly. Predictions for the absolute values of the quark masses are obtained e.g from QCD sum rules or lattice calculations. They depend on the running scale μ of the $\overline{\text{MS}}$ renormalization scheme. The modified minimal subtraction scheme $\overline{\text{MS}}$ will be used in section 2.8.3. The scale μ may e.g. be chosen such, that m_s equals the mass difference between isomultiplets differing by one unit of strangeness in some $SU(3)$ multiplet. The estimates for m_s range from about 120 MeV ($\mu = 1\,\text{GeV}$) up to 180 MeV ($\mu = 2\,\text{GeV}$).

In the following work it is presented, how the quark mass ratio

$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

may be determined by investigating the decay $\eta \to 3\pi$, using dispersion relations, married with chiral perturbation theory – and with the help of computers.
At treelevel, the χPT prediction for the rate of the decay $\eta \to \pi^+\pi^-\pi^0$ is $\Gamma^{\text{tree}} = 66\,\text{eV}$. The one-loop calculation leads to $\Gamma^{\text{1-loop}} = 160\,\text{eV}$. This is quite a large correction – the one-loop result accounts for contributions of $\mathcal{O}(m_s)$, which typically are of order 25% – nevertheless the one-loop prediction is still far away from the experimental value $\Gamma^{\text{exp}} = (273 \pm 26)\,\text{eV}$. One may expect higher loop calculations to enhance the prediction for the decay rate further. Taking more loops into account will give a better description of the final state interactions which are rather large in the present case. However, such calculations would get quite arduous. An elegant method which allows one to account for even large final state interaction effects are dispersion relations.
Using the leading order expressions for the masses of the pseudoscalar mesons (see

section 2.5) together with the Dashen theorem (section 2.6.2), one finds that Q is related to the physical kaon and pion masses by

$$\frac{1}{Q^2} = \frac{\left(m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2\right) m_{\pi^0}^2}{m_K^2 \left(m_K^2 - m_{\pi^0}^2\right)} ,$$

where $m_K^2 = \frac{1}{2}\left(m_{K^0}^2 + m_{K^+}^2\right)$. Inserting the experimental values for the masses into the above formula, we get $Q = 24.2$ as a standard value.
Now, the point is that the decay rate for $\eta \to 3\pi$ is proportional to the quark mass ratio Q^{-4} – we may turn the tables and use the prediction for the factor of proportionality to determine Q with the help of the experimental value Γ^{exp}.
Crudely speaking, unitarity will show us how to connect the imaginary part of the decay amplitude with the amplitude itself. The fact that an analytic function is uniquely characterized by its singularities will allow us to represent the amplitude by an integral over its discontinuities along a branch cut. These discontinuities are determined by the amplitude's imaginary part. Combining the implications of unitarity and analyticity will lead then to a couple of integral equations, whose solution can be found iteratively. Following this way will give us a more accurate representation of the discontinuities than the one-loop result. However, unitarity and analyticity will determine the amplitude only up to some subtraction polynomial – χPT will be needed to fix them. Quite a large part of the time I had at my disposal for the diploma work was spent to persuade the computer to solve the integral equations.
In principle, the dispersive analysis of $\eta \to 3\pi$ allows one to determine the quark mass ratio Q rather accurately – an uncertainty of 10% in the decay amplitude will reduce to a 2.5%–error in Q.

2 Physical aspects

2.1 The Noether theorem

In this section I will very briefly sketch some aspects of the Noether theorem that will be used in the next section. For a much more detailed exposé I refer to the textbooks [3] and [5].
We consider a theory with a set $\varphi(x) = \{\varphi_1(x), \ldots, \varphi_N(x)\}$ of fields and Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$. The action is defined as $S = \int d^4x\, \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$. The principle of least action, which is a classical one, says that the physical field configuration will be such that S is invariant under any infinitesimal variation $\delta\varphi$ of the fields φ, which does not hurt the conditions posed upon $\varphi(x)$. If we impose the natural requirement on $\varphi(x)$ to vanish as $x \to \infty$, the same has to be satisfied by $\delta\varphi(x)$. The variation of S due to $\delta\varphi(x)$ is given by:

$$\delta S = \sum_i \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} \delta\varphi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} \partial_\mu \delta\varphi_i \right) = 0.$$

Performing a partial integration and noting that the surface terms vanish, we get:

$$\delta S = \sum_i \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} \right) \delta\varphi_i(x) = 0.$$

Since this must hold for (almost) arbitrary $\delta\varphi(x)$, we deduce the Euler-Lagrange equations, which have to be satisfied by every dynamical field appearing in the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} = 0. \tag{2.1}$$

These are the classical equations of motion for the fields $\varphi_i(x)$.
We now turn to the Noether theorem. I will discuss it here only for the case of internal symmetries. Assume $\mathcal{L}$ to be invariant under the transformation $\varphi_i(x) \to \varphi_i(x) + i\epsilon F_i(x)$, where ϵ is an infinitesimal constant and $F_i(x)$ may depend on $\varphi(x)$ and $\partial_\mu\varphi(x)$. Independently of whether $\varphi(x)$ satisfies the equations of motion or not, this reads:

$$\delta\mathcal{L} = i\epsilon \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} F_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} \partial_\mu F_i(x) \right) = 0. \tag{2.2}$$

If we now in addition take $\varphi(x)$ to satisfy the Euler-Lagrange equations (2.1), we immediately find the current

$$J^\mu(x) = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} F_i(x) \tag{2.3}$$

to fulfill the continuity equation

$$\partial_\mu J^\mu = 0. \tag{2.4}$$

There is a conserved charge associated with each current:

$$Q = \int d^3x\, J^0(x)\,. \tag{2.5}$$

On the quantum mechanical level the charge Q becomes an operator. Supposed that $\partial_0 F_i = 0$, it acts as a generator of the corresponding symmetry, in the sense that $[\varphi_i(x), Q] = F_i(x)$ and an infinitesimal transformation of the fields $\varphi_i(x)$ may thus be written as

$$\varphi_i'(x) = \varphi_i(x) + i\epsilon[\varphi_i(x), Q].$$

I just mention that other conserved currents may be found due to external symmetries, coordinate transformations leaving the Lagrangian L (not necessarily the Lagrangian density $\mathcal{L}$) invariant. The most important consequences thereof are the conservation of energy, momentum and angular momentum.

2.2 The Goldstone theorem

Consider a Lagrangian $\mathcal{L}(\varphi_1, ..., \varphi_N, \partial_\mu\varphi_1, ..., \partial_\mu\varphi_N)$, where the φ_n are hermitian scalar fields. Suppose that the Lagrangian is left invariant under the global symmetry transformation $\varphi_n(x) \to \varphi_n(x) + i\epsilon \sum_m t_{nm}\varphi_m(x)$:

$$\delta\mathcal{L} = \sum_n \left(\frac{\partial\mathcal{L}}{\partial\varphi_n} i\epsilon \sum_m t_{nm}\varphi_m(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_n} i\epsilon \sum_m t_{nm}\partial_\mu\varphi_m(x) \right) = 0.$$

In the following we are interested in the physical vacuum. When considering the vacuum one would intuitively think the vacuum expectation value of the fields to vanish. However, this needs not to be true – the physical vacuum, being the lowest energy eigenstate, may happen to be a non-singlet state where the fields pick up an expectation value. What happens is, that the vacuum itself does not show the symmetry properties of the underlying theory, namely the corresponding Lagrangian. This phenomenon is called spontaneous symmetry breaking. We now want to consider the case where such a spontaneous symmetry breaking takes place, and show the consequences for the masses of the particles of the theory. The vacuum state is denoted by $\varphi^{\text{Vac}} = (\varphi_1^{\text{Vac}}, ..., \varphi_N^{\text{Vac}})$.
We make the important assumption that the vacuum is translation invariant. You may cast this in doubt: The spontaneous breakdown of a symmetry only takes place when the temperature, which after bigbang was arbitrary high, is low enough. One may think of the possibility that, during the universe was cooling down, different domains with a vacuum state of their own were formed. However, no hints of such a scenario influencing today's particle physics have been found and we may take translation invariance of the vacuum as granted:

$$\partial_\mu\varphi_n^{\text{Vac}} = 0.$$

The variation of $\mathcal{L}$ then reads:

$$\delta\mathcal{L}|_{\text{Vac}} = \sum_{n,m} \left.\frac{\partial\mathcal{L}}{\partial\varphi_n}\right|_{\text{Vac}} i\epsilon\, t_{nm}\varphi_m^{\text{Vac}} = 0.$$

From this, performing another derivative with respect to φ_l, we deduce:

$$\sum_{n,m} \left.\frac{\partial^2\mathcal{L}}{\partial\varphi_n\partial\varphi_l}\right|_{\text{Vac}} i\epsilon t_{nm}\varphi_m^{\text{Vac}} + \sum_n \left.\frac{\partial\mathcal{L}}{\partial\varphi_n}\right|_{\text{Vac}} i\epsilon t_{nl} = 0. \tag{2.6}$$

The Hamiltonian density is given by $\mathcal{H} = \sum_n \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_n} \partial_\mu \varphi_n - \mathcal{L}$. For the vacuum this means $\mathcal{H} = -\mathcal{L}$ and the condition for the vacuum to be a state of minimal energy reads:

$$\left.\frac{\partial \mathcal{H}}{\partial \varphi_n}\right|_{\varphi_n^{\mathrm{Vac}}} = -\left.\frac{\partial \mathcal{L}}{\partial \varphi_n}\right|_{\varphi_n^{\mathrm{Vac}}} = 0. \tag{2.7}$$

A state of minimal energy has to exist – otherwise the system would not be stable. Having a theory equipped with spontaneous symmetry breaking means:

$$\sum_m t_{nm} \varphi_m^{\mathrm{Vac}} \neq 0 \quad \text{for certain } n.$$

On the other hand, in view of (2.7) equation (2.6) becomes

$$\sum_{n,m} \left.\frac{\partial^2 \mathcal{L}}{\partial \varphi_n \partial \varphi_l}\right|_{\mathrm{Vac}} i\epsilon t_{nm} \varphi_m^{\mathrm{Vac}} = 0. \tag{2.8}$$

Introducing the matrix $\mathcal{M}^2$ with elements $\mathcal{M}^2{}_{ln} = -\frac{\partial^2 \mathcal{L}}{\partial \varphi_n \partial \varphi_l}$ and defining $\bar{\varphi}_n = \sum_m t_{nm} \varphi_m^{\mathrm{Vac}}$, we can write equation (2.8) as $\mathcal{M}^2 \bar{\varphi} = 0$.
Assume that we have chosen our field basis in such a way that the free part of $\mathcal{H}$ is diagonalized. $\mathcal{M}^2$ is nothing else than the mass matrix belonging to our Lagrangian and will be diagonal then, too. To conclude, we have

$$\begin{pmatrix} m_1^2 & & \\ & \ddots & \\ & & m_N^2 \end{pmatrix} \begin{pmatrix} \bar{\varphi}_1 \\ \vdots \\ \bar{\varphi}_N \end{pmatrix} = 0. \tag{2.9}$$

The fact that the vacuum is a state of minimal energy ensures $m_i^2 \geq 0$.
Let's denote the generators of the symmetry group G by t^a. An infinitesimal transformation of φ^{Vac} is then given by $\varphi^{\mathrm{Vac}} \to \alpha_a t^a \varphi^{\mathrm{Vac}}$. All the t^a with $t^a \varphi^{\mathrm{Vac}} = 0$ then generate a subgroup H of G. The generators $t^a \notin H$ generate the coset space G/H. The vectors $t^a \varphi^{\mathrm{Vac}} \neq 0$ are linearly independent, otherwise one could find a linear combination $T \neq 0$ of t^a's $\in G/H$ with $T\varphi^{\mathrm{Vac}} = 0$. The $|G| - |H|$ generators of G/H thus lead to $|G| - |H|$ linearly independent equations of the form $\mathcal{M}^2 t^a \varphi^{\mathrm{Vac}} = 0$. From this we conclude that there must be $|G| - |H|$ zero eigenvalues m_i^2.
Up to now the arguments were classical ones. There exists a similar quantum mechanical proof using the effective action description. However, to see that the Goldstone theorem holds also after quantization, let's look at an other argumentation.
According to the statement in the last section, we have

$$[\varphi_i(x), Q] = t_{ik} \varphi_k(x), \tag{2.10}$$

where Q denotes the conserved charge associated with the symmetry transformation $\varphi_i(x) \to \varphi_i(x) + i\epsilon t_{ik} \varphi_k(x)$. Instead of φ_i^{Vac} we now have to consider the vacuum expectation values $\langle 0|\varphi_i(x)|0\rangle$, which, assuming again translation invariance of the vacuum, are independent of x. If the symmetry is broken, there exists an i for which $t_{ik} \langle 0|\varphi_k(x)|0\rangle \neq 0$ and we find, in view of (2.10),

$$\langle 0| [\varphi_i(x), Q] |0\rangle = t_{ik} \langle 0|\varphi_k(x)|0\rangle \neq 0. \tag{2.11}$$

This clearly implies $Q|0\rangle \neq 0$. Inserting a complete set of momentum eigenstates, the vacuum expectation value of $[\varphi_i(x), Q]$ becomes

$$\langle 0|\left[\varphi_i, Q\right]|0\rangle = \int d^3y \sum_n \left\{\langle 0|\varphi_i|n\rangle \langle n|J^0(y)|0\rangle - \langle 0|J^0(y)|n\rangle \langle n|\varphi_i|0\rangle\right\} =$$

$$\sum_n \frac{\delta^3}{(2\pi)^3}(\vec{p}_n)\left\{e^{+ip_n^0 y^0}\langle 0|\varphi_i|n\rangle \langle n|J^0(0)|0\rangle - e^{-p_n^0 y^0}\langle 0|J^0(0)|n\rangle \langle n|\varphi_i|0\rangle\right\}, \quad (2.12)$$

where we used $J^\mu(x) = e^{iPx}J^\mu(0)e^{-iPx}$ and $P|0\rangle = 0$, $P|n\rangle = p_n|n\rangle$. The fact, that the current J is conserved, is not affected by spontaneous symmetry breaking. Therefore

$$\int_V d^3x \left\langle 0\left|\left[\varphi_i, \partial_0^y J^0(y)\right]\right|0\right\rangle = -\int_V d^3x \left\langle 0\left|\left[\varphi_i, \partial_i^y J^i(y)\right]\right|0\right\rangle =$$

$$-\int_{\partial V} d\vec{\sigma}\left\langle 0\left|\left[\varphi_i, \vec{J}(y)\right]\right|0\right\rangle \longrightarrow 0 \quad (V \to \infty). \quad (2.13)$$

This implies that

$$\sum_n p_n^0 \delta^3(\vec{p}_n)\left\{e^{+ip_n^0 y^0}\langle 0|\varphi_i|n\rangle \langle n|J^0(0)|0\rangle + e^{-p_n^0 y^0}\langle 0|J^0(0)|n\rangle \langle n|\varphi_i|0\rangle\right\} \quad (2.14)$$

vanishes for all y^0. The fact that the left hand side of equation (2.12) is different from zero, requires that there exists a state $|N\rangle$ for which $\langle 0|\varphi_i|N\rangle \langle N|J^0(y)|0\rangle \neq 0$. However, the expression (2.14) will then only vanish for all y^0 if $p_N^0 = 0$ for $\vec{p}_N = 0$ and we conclude that $|N\rangle$ has to be a massless state.
Since φ_i is a scalar field, $\varphi_i|0\rangle$ must be rotation invariant. For any state $|n\rangle$ with helicity different from zero, we therefore have $\langle 0|\varphi_i|n\rangle \equiv 0$. The same holds for $\langle n|J^0|0\rangle$ for states $|n\rangle$ with different unbroken internal quantum numbers than J^0. Thus $|N\rangle$ must be a state of spin zero and the quantum numbers of J^0.

2.3 The chiral limit

The Lagrangian of QCD with $N_f = 3$ is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{q}i\gamma^\mu D_\mu q + \bar{q}\mathcal{M}q,$$

where $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + f^{abc}G^b_\mu G^c_\nu$ is the gluon field strength. The color-gauge covariant derivative is given by $D_\mu = \partial_\mu - iG^a_\mu \frac{\lambda^a}{2}$. $\mathcal{M}$ denotes the quark mass matrix and q collects the quark fields:

$$\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}.$$

The right- and left-handed quark fields are defined as

$$q_R = \frac{1+\gamma_5}{2}q\,, \qquad \bar{q}_R = \bar{q}\frac{1-\gamma_5}{2}\,,$$
$$q_L = \frac{1-\gamma_5}{2}q\,, \qquad \bar{q}_L = \bar{q}\frac{1+\gamma_5}{2}\,,$$

according to their transformation properties under application of the operator γ_5:

$$\gamma_5 q_R = q_R\,, \quad \gamma_5 q_L = -q_L.$$

In the case of massless QCD, $\mathcal{M} = 0$, the right- and left-handed parts decouple (in view of $\gamma_5\gamma_5 = 1$ and $\{\gamma_5, \gamma^\mu\} = 0$) and the Lagrangian may be written as

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} + \bar{q}_R i\gamma^\mu D_\mu q_R + \bar{q}_L i\gamma^\mu D_\mu q_L.$$

Consider the transformation $q_R \to V_R q_R$, $q_L \to V_L q_L$, which implies $\bar{q}_R \to \bar{q}_R V_R^\dagger$, $\bar{q}_L \to \bar{q}_L V_L^\dagger$. Provided $V_R^\dagger V_R = V_L^\dagger V_L = 1$, the above Lagrangian is invariant under this transformation – we have found a global symmetry of massless QCD. Within classical chromo dynamics the symmetry group G is given by $U(3)_R \times U(3)_L = SU(3)_R \times U(1)_R \times SU(3)_L \times U(1)_L$. G is generated by totally $2N_f^2 = 18$ generators, each of them giving rise to a conserved Noether current. However, upon quantization one of them turns out not to be conserved; it will be specified below. In QCD, the symmetry group is reduced to $G = SU(3)_R \times SU(3)_L \times U(1)_{R+L}$.
The elements of $U(1)$ are of the form $\exp(i\alpha)$, whereas the elements of an $SU(3)$ are generated by $\vec{\lambda} = (\lambda^1, ..., \lambda^8)$; each $V \in SU(3)$ can be written as

$$V = \exp\left(i\sum_{a=1}^{8}\varphi^a\frac{\lambda^a}{2}\right) = \exp\left(i\vec{\varphi}\frac{\vec{\lambda}}{2}\right).$$

The operator $\frac{1+\gamma_5}{2}$, acting on q_R, has the same effect as the unity operator, but will give zero operating on q_L. Interchanging q_R with q_L, the analogous relations hold for the operator $\frac{1-\gamma_5}{2}$. We define the generators

$$\begin{aligned}\vec{\lambda}_R &= \frac{1+\gamma_5}{2}\vec{\lambda},\\ \vec{\lambda}_L &= \frac{1-\gamma_5}{2}\vec{\lambda}.\end{aligned}$$

Their commutators are:

$$\left[\frac{\lambda^a_R}{2}, \frac{\lambda^b_R}{2}\right] = if^{abc}\frac{\lambda^c_R}{2}, \quad \left[\frac{\lambda^a_L}{2}, \frac{\lambda^b_L}{2}\right] = if^{abc}\frac{\lambda^c_L}{2}, \quad \left[\frac{\lambda^a_R}{2}, \frac{\lambda^b_L}{2}\right] = 0.$$

Using these two sets of generators of $SU(3)_R$ and $SU(3)_L$, respectively, we have

$$\begin{aligned}V_R q_R &= \exp\left(i\vec{\varphi}_R\frac{\vec{\lambda}_R}{2}\right)q_R = \exp\left(i\vec{\varphi}_R\frac{\vec{\lambda}}{2}\right)q_R,\\ V_R q_L &= q_L,\\ V_L q_L &= \exp\left(i\vec{\varphi}_L\frac{\vec{\lambda}_L}{2}\right)q_L = \exp\left(i\vec{\varphi}_L\frac{\vec{\lambda}}{2}\right)q_L,\\ V_L q_R &= q_R.\end{aligned}$$

The two matrices V_R and V_L may thus be combined to a matrix V acting directly on q:

$$Vq = \exp\left(i\vec{\varphi}_R\frac{\vec{\lambda}_R}{2} + i\vec{\varphi}_L\frac{\vec{\lambda}_L}{2}\right)q.$$

V can be written in yet an other way, splitting the exponent into two pieces, one of them transforming q_R and q_L into the same "direction", the other describing the "difference" between V_R and V_L:

$$V = \exp\left(i\frac{\vec{\varphi}_R + \vec{\varphi}_L}{2}(\vec{\lambda}_R + \vec{\lambda}_L) + i\frac{\vec{\varphi}_R - \vec{\varphi}_L}{2}(\vec{\lambda}_R - \vec{\lambda}_L)\right) = \exp\left(i\vec{\varphi}^V\frac{\vec{\lambda}}{2} + i\vec{\varphi}^A\gamma_5\frac{\vec{\lambda}}{2}\right),$$

where $\vec{\varphi}^V = (\vec{\varphi}_R + \vec{\varphi}_L)$ and $\vec{\varphi}^A = (\vec{\varphi}_R - \vec{\varphi}_L)$. We have replaced the generators λ_R^a and λ_L^a of $SU(3)_R \times SU(3)_L$ by the generators $\lambda_V^a = \lambda^a$ and $\lambda_A^a = \gamma_5\lambda^a$ with commutators

$$\left[\frac{\lambda_V^a}{2}, \frac{\lambda_V^b}{2}\right] = if^{abc}\frac{\lambda_V^c}{2}, \quad \left[\frac{\lambda_A^a}{2}, \frac{\lambda_A^b}{2}\right] = if^{abc}\frac{\lambda_V^c}{2}, \quad \left[\frac{\lambda_V^a}{2}, \frac{\lambda_A^b}{2}\right] = if^{abc}\frac{\lambda_A^c}{2}.$$

It was somehow a detour to reach the final form of V, which could have been seen directly. However, I think it's more instructive to follow the above way in order to see why this symmetry is called chiral symmetry – for $m_u = m_d = m_s = 0$ the QCD-Lagrangian for $N_f = 3$ is invariant under independent rotations of the left- and right-handed quark fields.
An infinitesimal symmetry transformation of the quark field q is given by:

$$q \to \left(1 + i\epsilon_V^a \frac{\lambda^a}{2} + i\epsilon_A^a \gamma_5 \frac{\lambda^a}{2} + i\alpha_{R+L}\right) q.$$

Using $\frac{\partial \mathcal{L}}{\partial \partial_\mu q} = i\bar{q}\gamma^\mu$, the Noether currents associated with the symmetry group G become, according to the definition given by equation (2.3):

$$\begin{aligned} V_\mu^a &= \bar{q}\gamma_\mu \frac{\lambda^a}{2} q\,, \\ A_\mu^a &= \bar{q}\gamma_\mu\gamma_5 \frac{\lambda^a}{2} q\,, \\ V_\mu^0 &= \bar{q}\gamma_\mu q\,. \end{aligned}$$

As already mentioned above, in the classical limit there is a 18th conserved current. It is associated with the symmetry transformation $q \to (1 + i\alpha_{R-L}\gamma_5)\, q$ and is given by $A_\mu^0 = \bar{q}\gamma_\mu\gamma_5 q$.
So far we have considered the case of massless QCD (with $N_f = 3$). Turning on the quark masses will couple the left- and righthanded quarkfields – chiral symmetry is explicitly broken. Since m_u, m_d and m_s are small, the corresponding symmetry breaking terms may be considered as a small perturbation in the QCD Hamiltonian – chiral symmetry is an approximate symmetry. The divergences of the currents V_μ^a and A_μ^a no longer vanish:

$$\begin{aligned} \partial^\mu V_\mu^a &= i\bar{q}\left[\mathcal{M}, \frac{\lambda^a}{2}\right] q\,, \\ \partial^\mu A_\mu^a &= i\bar{q}\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\} q\,. \end{aligned}$$

The divergences one immediately gets in view of the Dirac equation which holds for every color component separately: $(iD_\mu\gamma^\mu - m)q_c = 0$, where q_c denotes a quark spinor of flavour q and color c. In the case $m_u = m_d = m_s$ the commutator $[\mathcal{M}, \lambda^a/2]$ vanishes and V_μ^a is still a conserved current.
Besides this explicit symmetry breaking QCD is generally assumed to exhibit also a spontaneous symmetry breakdown, namely of the group $SU(3)_R \times SU(3)_L$ to its subgroup $SU(3)_{R+L}$, generated by the generators λ_V^a. In the case of massless QCD one therefore expects the theory to contain $3^2 - 1 = 8$ Goldstone bosons. The effect of the quark masses is, to turn these Goldstone bosons into "pseudo" Goldstones, no longer being massless particles. Indeed, the physical spectrum of QCD contains particles that very well fit this pattern known as the famous eightfold way: the pseudoscalar octet, consisting of the mesons ($\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η). Their quantum numbers are precisely those of the Noether currents A_μ^a associated with the spontaneous breakdown of chiral symmetry. Three of these mesons (the pions)

are particularly light, what is to be expected, if the group $SU(2)_R \times SU(2)_L$ is an almost exact symmetry leading to $2^2 - 1 = 3$ especially light pseudo Goldstones. This requires one of the quark masses m_u, m_d or m_s to be big compared to the other two. Since the pions carry no strangeness, it's m_s to be the big one.
In the case of vanishing quark masses the currents A^a_μ $(a = 1, \ldots, 8)$ are conserved. The charges Q^a associated with A^a_μ commute with the Hamiltonian H: $[Q^a, H] = 0$. Thus the state $Q^a|h\rangle$ has the same energy as the one-particle state $|h\rangle$, but is of opposite parity. If chiral symmetry were unbroken, $Q^a|h\rangle$ would be a one-particle state, too, degenerate with $|h\rangle$. However, no such parity doubling is observed – instead there exists a state of almost the same energy and opposite parity as $|h\rangle$ containing one hadron and a pion. This further corroborates the interpretation of the pseudoscalars as the Goldstone bosons of a spontaneously broken approximate symmetry.

2.4 Effective Lagrangian and χPT

The relevant degrees of freedom of a theory depend very much on the energy scale. The idea of effective field theories is, to find a description more appropriate to low energies than the underlying fundamental theory provides. The remnants of the underlying theory are its symmetries and the low energy constants which so to speak encapsulate the short-distance structure.
In principle the dynamics of the pseudoscalar mesons are determined by QCD, as they are bound states of this theory. However, up to now it isn't possible to solve QCD exactly. Moreover, due to the fact that color is confined, quarks and gluons don't appear as free physical states.
The effective field theory of QCD is called chiral perturbation theory (χPT). The quark and gluon fields are replaced by a set of eight pseudoscalars, the Goldstone bosons related to the spontaneous breakdown of $SU(3)_R \times SU(3)_L$ to $SU(3)_{R+L}$. We collect these pseudoscalars in the field $\vec{\pi} = \left(\pi^1, \ldots, \pi^8\right)$.
Let's denote the space of pion field variables by Π and its origin by 0_π. The neutral element of the symmetry group G we call 1_G. The pion field transforms according to a representation of the symmetry group G – this means that there exists a mapping

$$f: \; G \times \Pi \longrightarrow \Pi$$

such that

$$\begin{aligned} \forall\, \vec{\pi} \in \Pi: \;\; & f(1_G, \vec{\pi}) = \vec{\pi}, \\ \forall\, g_1,\, g_2 \in G,\, \vec{\pi} \in \Pi: \;\; & f(g_1 g_2, \vec{\pi}) = f\left(g_1, f(g_2, \vec{\pi})\right). \end{aligned} \tag{2.15}$$

This implies that those elements $h \in G$ with $f(h, 0_\pi) = 0_\pi$ form a subgroup $H \subset G$. We may define an equivalence relation $\sim$ on G by calling two elements $g_1,\, g_2 \in G$ equivalent exactly if $f(g_1, 0_\pi) = f(g_2, 0_\pi)$. From (2.15) one immediately gets

$$g \sim gh \qquad \forall\, h \in H,$$

and, noting $f\left(g_1^{-1}, f(g_1, 0_\pi)\right) = f(1_G, 0_\pi) = 0_\pi$, one finds

$$f(g_1, 0_\pi) = f(g_2, 0_\pi) \iff g_1^{-1} g_2 \in H.$$

Therefore the equivalence classes defined through the relation $\sim$ may be identified with the cosets $\{gh,\, h \in H\}$, the elements of G/H. We choose a representator n in each coset. There is a one to one mapping between the set N of representators and $\{f(g, 0_\pi),\; g \in G\}$, the values of the pion field variables, i.e. $\forall\, g \in G \;\; \exists$ exactly

one $n \in N$ such that $f(g, 0_\pi) = f(n, 0_\pi)$. The transformation law of the pion field is fully fixed. The action of $g \in G$ on $\vec{\pi}_n = f(n, 0_\pi)$ is:

$$\vec{\pi}'_n = f(g, \vec{\pi}_n) = f(g, f(n, 0_\pi)) = f(gn, 0_\pi) = f(n'h, 0_\pi) = \vec{\pi}_{n'}\,, \qquad h \in H.$$

In the present case $G = SU(3)_R \times SU(3)_L$ and $H = SU(3)_{R+L}$. The set of representators we choose to be $N = \{(V, 1), V \in SU(3)\}$. The coset $C \ni (V, U)$ is then represented by $(VU^\dagger, 1)$. Multiplication of two elements of G/H we define by multiplication of their representators $(V_1, 1), (V_2, 1) \in N$: $(V_1, 1)(V_2, 1) = (V_1 V_2, 1)$. The structure of the coset space G/H is thus the one of the manifold $SU(3)$. Note, that the diagram

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \downarrow & & \downarrow \\ N \times N & \longrightarrow & N \end{array}$$

does not commute, however.
The eight ordinary fields π^1, ..., π^8 may be collected in a matrix field $U \in SU(3)$, replacing $\vec{\pi}$. In this work I will stick to canonical coordinates:

$$U(x) = \exp\left(i\alpha\phi(x)\right) = \exp\left(i\alpha \sum_{i=1}^{8} \lambda_i \pi^i(x)\right), \tag{2.16}$$

where λ_i denote the Gell-Mann matrices. Using these coordinates allows us to embed the set Π in $SU(3)_R \times SU(3)_L$ by identifying U with the representator $(U, 1)$. The action of $g = (V_R, V_L) \in G$ on U we get from

$$gn = (V_R, V_L)(U, 1) = (V_R U, V_L) = (V_R U V_L^\dagger, 1)(V_L, V_L) = n'h.$$

so that the transformation law of the pion field reads

$$U'(x) = V_R U(x) V_L^\dagger. \tag{2.17}$$

We have considered here the case $N_f = 3$. In the general case the fields of the Goldstone bosons are collected in a $N_f \times N_f$ matrix field $\in SU(N_f)$. However, the mass terms associated with the heavy quarks (c, b, t) may not be considered as a small perturbation. Turning the masses m_c, m_b, m_t off would not be a good approximation. One therefore limits N_f to $N_f \leq 3$.
Using canonical coordinates the QCD-Lagrangian is replaced by an effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots)\,,$$

For $\mathcal{L}_{\text{eff}}$ we write down an expansion in powers of derivatives of the field. Lorentz invariance restricts the series to terms containing even powers of derivatives. For massless QCD this reads

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)}_{\text{eff}} + \mathcal{L}^{(2)}_{\text{eff}} + \mathcal{L}^{(4)}_{\text{eff}} + \mathcal{L}^{(6)}_{\text{eff}} + \ldots\,,$$

where the superscript n in $\mathcal{L}^{(n)}_{\text{eff}}$ denotes the number of derivatives appearing in the corresponding term. The form of $\mathcal{L}_{\text{eff}}$ is very much constraint, because it has to respect chiral symmetry. In the case $\mathcal{M} = 0$ the momentum independent term $\mathcal{L}^{(0)}_{\text{eff}}$ has to be a constant: setting $V_L = U$ and $V_R = 1$ we get

$$\mathcal{L}^{(0)}_{\text{eff}}(U) = \mathcal{L}^{(0)}_{\text{eff}}(1UU^\dagger) = \mathcal{L}^{(0)}_{\text{eff}}(1).$$

This reflects the fact that Goldstone Bosons with vanishing momenta don't interact. At $\mathcal{O}\left(p^2\right)$ one finds a single invariant. Dropping the irrelevant cosmological constant $\mathcal{L}_{\text{eff}}^{(0)}$, the leading contribution to the Lagrangian can be written in the form

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{F^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger \right\rangle, \tag{2.18}$$

where $\langle \mathcal{A} \rangle$ denotes the trace of the matrix $\mathcal{A}$. The expansion of the matrix field U reads:

$$U = 1 + i\alpha\phi - \frac{\alpha^2}{2}\phi^2 - \frac{i\alpha^3}{6}\phi^3 + \frac{\alpha^4}{24}\phi^4 + \mathcal{O}(\phi^5).$$

The kinetic part of the Lagrangian becomes $\mathcal{L}_{\text{kin}} = \frac{F^2\alpha^2}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}$. To get the standard normalization of this term, we have to set $\alpha = F^{-1}$.
The pion decay constant F_π is defined through the matrix element of the axial current A_μ^a (see preceding section) between the vacuum and a one-particle state:

$$\left\langle 0|A_\mu^a|\pi^b(p)\right\rangle = ip_\mu\delta^{ab}F_\pi. \tag{2.19}$$

The Noether currents of $\mathcal{L}_{\text{eff}}^{(2)}$ associated with the $SU(3)_R$ and the $SU(3)_L$ symmetry, respectively, are

$$J^a_{\mu\,R} = i\alpha\left\langle \lambda^a \partial_\mu U U^\dagger \right\rangle \quad \text{and} \quad J^a_{\mu\,L} = -i\alpha\left\langle \lambda^a U^\dagger \partial_\mu U \right\rangle.$$

From this we find the following expressions for the vector and axial currents:

$$\begin{aligned} V_\mu^a &= J^a_{\mu\,R} + J^a_{\mu\,L} = i\alpha\left\langle \lambda^a \left[\partial_\mu U, U^\dagger\right]\right\rangle, \\ A_\mu^a &= J^a_{\mu\,R} - J^a_{\mu\,L} = i\alpha\left\langle \lambda^a \left\{\partial_\mu U, U^\dagger\right\}\right\rangle. \end{aligned} \tag{2.20}$$

We evaluate the matrix element defining F_π using this explicit representation for A_μ^a:

$$\left\langle 0|A_\mu^a|\pi^b(p)\right\rangle = -\frac{F}{2}\left\langle 0\right|\left\langle \lambda^a\lambda_i\partial_\mu\pi^i\right\rangle\left|\pi^b(p)\right\rangle = -F\delta_i^a\left\langle 0|\partial_\mu\pi^i|\pi^b(p)\right\rangle = iF\delta^{ab}p_\mu,$$

and conclude $F = F_\pi$.
As already mentioned, chiral symmetry is explicitly broken by the quark masses. This leads to symmetry breaking contributions to $\mathcal{L}_{\text{eff}}$; the leading term is

$$\mathcal{L}_{\text{sb}} = \frac{F_\pi^2}{4}\left\langle \chi U^\dagger + U\chi^\dagger \right\rangle, \quad \text{where} \quad \chi = 2\mathcal{M}B.$$

The low energy constant B is real, if one requires the effective Lagrangian to be parity invariant and uses a diagonal and real quark mass matrix. For a derivation of the form of the symmetry breaking contribution see [12]. To maintain chiral bookkeeping, $\mathcal{M}$ has to be counted as two powers of momentum. Hence the leading term in $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{F_\pi^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger + 2\mathcal{M}B(U + U^\dagger)\right\rangle, \tag{2.21}$$

The theory described by $\mathcal{L}_{\text{eff}}^{(2)}$ is also referred to as the non-linear σ-model. Taking symmetry breaking into account, chiral perturbation theory results in a simultaneous expansion in momenta and quark masses. The expansion of $\mathcal{L}_{\text{eff}}^{(2)}$ reads

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} + \left\langle \frac{1}{48F_\pi^2}[\partial_\mu\phi,\phi][\partial^\mu\phi,\phi]\right\rangle + \\ \left\langle B\mathcal{M}F_\pi^2 - \frac{B}{2}\mathcal{M}\phi^2 + \frac{B}{24F_\pi^2}\mathcal{M}\phi^4\right\rangle + \mathcal{O}(\phi^6). \end{aligned} \tag{2.22}$$

The term $\langle B\mathcal{M}F_\pi^2\rangle = BF_\pi^2(m_u+m_d+m_s)$ is related to the quark condensates, the vacuum expectation values of the operators $\bar{q}q$. The operator $\bar{u}u$ is related to the QCD-Hamiltonian by $\bar{u}u = \frac{\partial H_{\rm QCD}}{\partial m_u}$. The vacuum expectation value of $\bar{u}u$ is thus equal to the derivative of the vacuum energy with respect to m_u. At leading order, the vacuum energy is $-\langle B\mathcal{M}F_\pi^2\rangle$ and we get

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle = -BF_\pi^2 + \mathcal{O}(m).$$

The quark mass condensates are called order parameters – they are a measure for the strength of the spontaneous symmetry breaking.
Finally let's write down the matrix ϕ in terms of the conventionally normalized pseudoscalar meson fields

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2), \qquad K^\pm = \frac{1}{\sqrt{2}}(\pi^4 \mp i\pi^5),$$
$$K^0 = \frac{1}{\sqrt{2}}(\pi^6 - i\pi^7), \qquad \bar{K}^0 = \frac{1}{\sqrt{2}}(\pi^6 + i\pi^7).$$

This yields

$$\phi = \begin{pmatrix} \pi^3 + \frac{1}{\sqrt{3}}\pi^8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^3 + \frac{1}{\sqrt{3}}\pi^8 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & \frac{2}{\sqrt{3}}\pi^8 \end{pmatrix}. \tag{2.23}$$

2.5 Masses of the eight lightest pseudoscalar mesons and $\eta\pi^0$-mixing

The symmetry breaking contributions to $\mathcal{L}_{\rm eff}$ provide mass terms for the pseudoscalar mesons. Expanding U, one may work out the kinetic part. At leading order this yields:

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm kin} \;=\; & \tfrac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} \;-\; \tfrac{B}{2}\mathcal{M}\langle\phi^2\rangle \;= \\
\partial_\mu\pi^+\partial^\mu\pi^- \;&-\; B(m_u+m_d)\pi^+\pi^- \;+ \\
\tfrac{1}{2}\partial_\mu\pi^3\partial^\mu\pi^3 \;&-\; \tfrac{1}{2}B(m_u+m_d){\pi^3}^2 \;+ \\
\partial_\mu K^+\partial^\mu K^- \;&-\; B(m_u+m_s)K^+K^- \;+ \\
\partial_\mu K^0\partial^\mu \bar{K}^0 \;&-\; B(m_d+m_s)\bar{K}^0K^0 \;+ \\
\tfrac{1}{2}\partial_\mu\pi^8\partial^\mu\pi^8 \;&-\; \tfrac{1}{2}B(\tfrac{m_u}{3}+\tfrac{m_d}{3}+\tfrac{4m_s}{3}){\pi^8}^2 \;+ \\
&-\; \tfrac{2B}{\sqrt{3}}\pi^3\pi^8(m_u-m_d)\,.
\end{aligned} \tag{2.24}$$

From this we read off the following leading terms to the mass squares of the charged pions and of the kaons:

$$\begin{aligned}
m^2_{\pi^\pm} &= B(m_u+m_d)\,, \\
m^2_{K^\pm} &= B(m_u+m_s)\,, \\
m^2_{K^0,\bar{K}^0} &= B(m_d+m_s)\,.
\end{aligned} \tag{2.25}$$

The appearance of the term $\frac{2B}{\sqrt{3}}\pi^3\pi^8(m_u-m_d)$ in equation (2.24), is an effect caused by isospin-symmetry breaking. The presence of this contribution mixed in $\pi^3\pi^8$ just tells us that the field basis we were working with so far does not correspond to the physical particles of our theory; namely that the π^3 field will not only create neutral pions but also η- particles. Thus we are forced to look for a new basis in order do get a diagonal mass matrix for the physical pseudoscalar mesons.
We reexpress π^3 and π^8 by the physical fields π^0 and η as follows:

$$\begin{aligned} \pi^3 &= \cos\epsilon\ \pi^0 - \sin\epsilon\ \eta, \\ \pi^8 &= \sin\epsilon\ \pi^0 + \cos\epsilon\ \eta. \end{aligned} \tag{2.26}$$

where ϵ denotes the mixing angle, which is small.
Using the approximation $\cos\epsilon \approx 1$ and $\sin\epsilon \approx \epsilon$, collecting all terms in $\pi^0\eta$ and requiring them to vanish, we get ϵ to leading order:

$$\epsilon = \frac{\sqrt{3}}{4}\frac{m_d - m_u}{m_s - \hat{m}}\,, \tag{2.27}$$

where $\hat{m} \equiv (m_u + m_d)/2$.
Again we work out the kinetic part of $\mathcal{L}_{\text{eff}}^{(2)}$ containing the π^0- and η-field. Up to order $(m_d - m_u)^2$ we get for the corresponding masses of the particles π^0 and η:

$$\begin{aligned} m_{\pi^0}^2 &= B(m_u + m_d) - \Delta\,, \\ m_\eta^2 &= B\left(\frac{4\,m_s + m_u + m_d}{3}\right) + \Delta\,, \\ \Delta &= \frac{2\epsilon}{\sqrt{3}}(m_u - m_d) + \epsilon^2\left(\frac{4\,m_s + m_u + m_d}{3} - (m_u + m_d)\right). \end{aligned}$$

Using the expressions for the mass squares for the $\pi^\pm$- and K-particles found above we may reexpress Δ and finally get:

$$\begin{aligned} m_{\pi^0}^2 &= m_{\pi^\pm}^2 - \frac{4}{3}\epsilon^2(m_K^2 - m_{\pi^\pm}^2), \\ m_\eta^2 &= B\left(\frac{4m_s + m_u + m_d}{3}\right) + \frac{4}{3}\epsilon^2(m_K^2 - m_{\pi^\pm}^2), \end{aligned} \tag{2.28}$$

where $m_K^2 = \frac{1}{2}\left(m_{K^+}^2 + m_{K^0}^2\right)$.
The $\pi^0\eta$-mixing causes a difference between $m_{\pi^\pm}^2$ and $m_{\pi^0}^2$. The masses of the charged pions and the kaons are not affected by the mixing. From equation (2.25) and (2.28) one also reads off the Gell-Mann-Okubo formula which holds to leading order in the quark mass expansion:

$$3m_\eta^2 + m_\pi^2 = 4m_K^2. \tag{2.29}$$

2.6 The transition amplitude $A_{\eta\to\pi^+\pi^-\pi^0}$

2.6.1 Current algebra result

To calculate the transition amplitude $A_{\eta\to\pi^+\pi^-\pi^0}$ at $\mathcal{O}(p^2)$ we have to extract the terms of $\mathcal{L}_{\text{eff}}^{(2)}$ containing one η-, one π^+-, one π^-- and one π^0-field or derivatives thereof. From equation (2.22) we obtain two parts relevant for the $\mathcal{O}(p^2)$ interaction Lagrangian $\mathcal{L}_{\eta\to3\pi}^{(2),tree}$:

$$\frac{B}{3\sqrt{3}F_\pi^2}(m_u - m_d)\,\eta\,\pi^+\pi^-\pi^0 \tag{2.30}$$

arising from $\frac{B}{24F_\pi^2}\langle\mathcal{M}\phi^4\rangle$ and

$$\frac{1}{3F_\pi^2}\epsilon(2\partial_\mu\pi^-\partial^\mu\pi^+\pi^0\eta + 2\partial_\mu\pi^0\partial^\mu\eta\pi^+\pi^- - \partial_\mu\pi^-\partial^\mu\eta\pi^0\pi^+ - \partial_\mu\pi^+\partial^\mu\eta\pi^0\pi^- - \partial_\mu\pi^-\partial^\mu\pi^0\eta\pi^+ - \partial_\mu\pi^+\partial^\mu\pi^0\eta\pi^-) \quad (2.31)$$

from expanding $\frac{1}{48F_\pi^2}\langle[\partial_\mu\phi,\phi][\partial^\mu\phi,\phi]\rangle$. In this approximation, the transition amplitude is given by $\langle\pi^+\pi^-\pi^0|\mathcal{L}^{(2),tree}_{\eta\to3\pi}|\eta\rangle$. Using creation and annihilation operators we get the usual Fourier decomposition for the field operators, namely

$$\varphi_{\text{neutral}}(x) = \int\frac{d^3p}{(2\pi)^3 2p_0}\left(a^\dagger_\varphi(\vec{p})e^{ipx} + a_\varphi(\vec{p})e^{-ipx}\right)$$

for neutral fields (as η or π^0) and

$$\varphi_{\text{charged}}(x) = \int\frac{d^3p}{(2\pi)^3 2p_0}\left(b^\dagger_\varphi(\vec{p})e^{ipx} + c_\varphi(\vec{p})e^{-ipx}\right)$$

for charged fields (as π^+ or π^-). The field operators for particles and antiparticles are the hermitian conjugate of each other. Further we have $[a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3 2p_0\delta^3(\vec{p}-\vec{p}\,')$, where a and $a^\dagger$ stand for any annihilation- and creation operator, respectively. All other commutators vanish. The states occurring here are

$$\begin{aligned}|\eta(p_\eta)\rangle &= a^\dagger_\eta(p_\eta)|0\rangle,\\ |\pi^+(p_+),\pi^-(p_-),\pi^0(p_0)\rangle &= b^\dagger_{\pi^\pm}(p_+)c^\dagger_{\pi^\pm}(p_-)a^\dagger_{\pi^0}(p_0)|0\rangle.\end{aligned}$$

When contracting the operators according to Wick's theorem, one gets some δ-functions which can be integrated out. What remains is a factor ip^μ_π for each $\partial^\mu\pi$ and a factor $-ip^\mu_\eta$ for $\partial^\mu\eta$. The term containing no derivatives yields

$$\frac{B}{3\sqrt{3}F_\pi^2}(m_u - m_d). \quad (2.32)$$

The other term gives a contribution a little bit more complicated:

$$-\frac{\epsilon}{3F_\pi^2}(2p_-p_+ - 2p_0p_\eta + p_-p_\eta + p_+p_\eta - p_-p_0 - p_+p_0). \quad (2.33)$$

Using $p_\eta = p_+ + p_- + p_0$ and adding the two contributions yields, up to $\mathcal{O}(\epsilon)$,

$$A_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{3F_\pi^2}\Big(\frac{B}{\sqrt{3}}(m_u - m_d) - \epsilon\left(p_+^2 + p_-^2 - 2p_0^2 + 4p_+p_- - 2p_+p_0 - 2p_-p_0\right)\Big) \quad (2.34)$$

Further we have from equation (2.25) and (2.27)

$$B(m_u - m_d) = -\frac{4\epsilon B}{\sqrt{3}}\left(m_s - \frac{m_u + m_d}{2}\right) = -\frac{\epsilon B}{\sqrt{3}}(4m_s + m_u + m_d - 3(m_u + m_d)) = -\sqrt{3}\epsilon(m_\eta^2 - m_\pi^2). \quad (2.35)$$

At last we make use of $p^2_{+,-,0} = m_\pi^2$ and $(p_+ + p_- + p_0)^2 = m_\eta^2$ to get the final result for the transition amplitude in the treelevel approximation:

$$A_{\eta\to\pi^+\pi^-\pi^0} = -\frac{\epsilon}{F_\pi^2}\left(s - \frac{4}{3}m_\pi^2\right), \quad (2.36)$$

where s is one of the three Mandelstam variables: $s=(p_+ + p_-)^2$, $t=(p_- + p_0)^2$ and $u=(p_0+p_+)^2$. s, t and u are related by

$$s+t+u = m_\eta^2 + 3m_\pi^2 = 3s_0. \tag{2.37}$$

The $\mathcal{O}(p^2)$ chiral perturbation theory calculation (2.36) reproduces the current algebra result.

2.6.2 Electromagnetic corrections

At leading order in the low energy expansion, the electromagnetic interaction can be described by an effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}}^{(2),em} = C\cdot\left\langle \mathcal{Q}U\mathcal{Q}U^\dagger\right\rangle , \tag{2.38}$$

where $\mathcal{Q}$ is the charge matrix of the three lightest quarks:

$$\mathcal{Q} = \frac{e}{3}\begin{pmatrix} 2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{2.39}$$

In order to maintain the usual chiral counting, one may set $e\sim\mathcal{O}(p)$. Thus (2.38) indeed yields a Lagrangian of $\mathcal{O}(p^2)$.
The term quadratic in ϕ,

$$\frac{C}{2F_\pi^2}\left\langle[\mathcal{Q},\phi]^2\right\rangle = -\frac{2Ce^2}{F_\pi^2}\left(\pi^+\pi^- + K^+K^-\right),$$

accounts for the self-energy of $\pi^\pm$ and $K^\pm$. It contributes equally to their mass squares, which implies,

$$\left(m_{K^0}^2 - m_{K^+}^2\right)_{\text{QCD}} = m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2.$$

This is Dashen's theorem. χPT provides an elegant derivation thereof.
The electromagnetic interaction causes the main part of the difference $m_{\pi^\pm}^2 - m_{\pi^0}^2$ – it is much bigger than the $\mathcal{O}\left((m_d-m_u)^2\right)$ contribution stemming from $\eta\pi^0$-mixing (see equation(2.28)).

$$\left(m_{\pi^\pm}^2 - m_{\pi^0}^2\right)_{em} = \frac{2Ce^2}{F_\pi^2} + \mathcal{O}\left(me^2\right). \tag{2.40}$$

The terms in $\mathcal{L}_{\text{eff}}^{(2),em}$ quartic in ϕ,

$$\frac{1}{F_\pi^2}\left\langle \frac{1}{12}\mathcal{Q}^2\phi^4 - \frac{1}{3}\mathcal{Q}\phi\mathcal{Q}\phi^3 + \frac{1}{4}\mathcal{Q}\phi^2\mathcal{Q}\phi^2\right\rangle,$$

give no contribution of the form $\pi^+\pi^-\pi^0\eta$. At treelevel, $\mathcal{L}_{\text{eff}}^{(2),em}$ therefore induces no transition $\eta\to 3\pi$. This is no longer true when taking one-loop graphs into account – there will arise $\mathcal{O}(me^2)$ contributions.
The current algebra expression, equation (2.36), is modified because of the self energy of $\pi^\pm$. Equation (2.35) remains unchanged ($B(m_u+m_d)=m_\pi^2$). Reevaluating the part of (2.33) involving the pion momenta changes the current algebra result to

$$A_{\eta\to\pi^+\pi^-\pi^0} = -\frac{\epsilon}{F_\pi^2}\left(s - \frac{2}{3}\left(m_{\pi^+}^2 + m_{\pi^0}^2\right)\right). \tag{2.41}$$

On the one-loop level there will also be final state Coulomb interactions between the charged pions. It leads to an $\mathcal{O}\left(e^2p^2(m_d-m_u)/(m_s-\hat{m})\right)$ contribution to the decay amplitude. For a discussion of electromagnetic corrections to the decay $\eta\to 3\pi$ of order p^2e^2 see [8] – this calculation shows that the electromagnetic corrections are negligibly small.

2.6.3 General structure of the transition amplitude

The main contribution to the decay amplitude $A_{\eta\to 3\pi}$ is due to the isospin breaking part $H_1 = \frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)$ of the QCD Hamiltonian. So, at leading order in $(m_u - m_d)$, the decay amplitude is given by

$$A_{\eta\to 3\pi} = \frac{m_u - m_d}{2}\langle\pi^i\pi^k\pi^l|\bar{u}u - \bar{d}d|\eta\rangle = \frac{m_u - m_d}{2}\langle\pi^i\pi^k\pi^l|\bar{q}\lambda^3 q|\eta\rangle.$$

Since the pions, having isospin $I = 1$, form an isotriplet, they transform under isospin rotations according to the representation D^1:

$$\pi^i \to R^{ik}\pi^k,$$

where $R \in O(3)$.
The behaviour of an operator $\bar{q}\lambda^m q$ under a transformation $q \to Uq$, $U \in SU(3)$ is $\bar{q}\lambda^m q \to \bar{q}U^\dagger\lambda^m U q$. In the case of pure isospin-rotations the matrix U is restricted to matrices of the form $\left(\begin{smallmatrix} V & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $V \in SU(2)$, and $\lambda^m \in \{\lambda^1, \lambda^2, \lambda^3\}$. From $\langle U^\dagger\lambda^i U\rangle = \langle\lambda^i\rangle = 0$ one concludes that $U^\dagger\lambda^i U = R^{ik}\lambda^k$. Further we have:

$$\langle U^\dagger\lambda^i U U^\dagger\lambda^k U\rangle = \langle\lambda^i\lambda^k\rangle = 2\delta^{ik}\,,$$

on the other hand

$$\langle U^\dagger\lambda^i U U^\dagger\lambda^k U\rangle = \langle R^{il}\lambda^l R^{km}\lambda^m\rangle = 2\delta^{lm}R^{il}R^{km}\,,$$

and so

$$R^{il}R^{kl} = (RR^T)^{ik} = \delta^{ik} \quad \Rightarrow \quad RR^T = \mathbf{1}.$$

Any isospin rotation can therefore be described through a matrix $R \in O(3)$. The matrix element $A_{\eta\to 3\pi}$ is invariant under such a transformation which implies

$$\langle\pi^a\pi^b\pi^c|\bar{q}\lambda^d q|\eta\rangle = R^{ia}R^{kb}R^{lc}R^{md}\langle\pi^i\pi^k\pi^l|\bar{q}\lambda^m q|\eta\rangle. \tag{2.42}$$

Now consider a function f depending on the four three component vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ and $\vec{v}$. Suppose f to be invariant under transformation of its arguments by any matrix $R \in O(3)$. In that case one may show that f only depends on scalar products formed out of the four vectors. (E.g. we may, without loss of generality, set $\vec{v} = (0, 0, v)$ - so $v^2 = \vec{v}\cdot\vec{v}$. Similar reasoning for the components of the other vectors leads to the desired statement.)
Set $x^i = R^{ia}$, $y^k = R^{kb}$, $z^l = R^{lc}$ and $v^m = R^{md}$ and define

$$f(\vec{x},\vec{y},\vec{z},\vec{v}) = \sum_{i,k,l,m=1}^{3} x^i y^k z^l v^m \langle\pi^i\pi^k\pi^l|\bar{q}\lambda^m q|\eta\rangle,$$

which is nothing else than the righthand side of equation (2.42). Performing a second isospin rotation (by R) now amounts in a rotation of the arguments of f (by R^T) which is invariant under this operation. We may now write:

$$f(\vec{x},\vec{y},\vec{z},\vec{v}) = A(\vec{x}\cdot\vec{y})(\vec{z}\cdot\vec{v}) + B(\vec{x}\cdot\vec{z})(\vec{y}\cdot\vec{v}) + C(\vec{x}\cdot\vec{v})(\vec{y}\cdot\vec{z}).$$

For the decay amplitude this means:

$$\begin{aligned}\langle\pi^i(\vec{p}_1)\pi^k(\vec{p}_2)\pi^l(\vec{p}_3)|\bar{q}\lambda^m q|\eta(\vec{p}_\eta)\rangle = i(2\pi)^4\delta^4(p_1 + p_2 + p_3 - p_\eta)\\ A(s,t,u)\delta^{ik}\delta^{lm} + B(s,t,u)\delta^{il}\delta^{km} + C(s,t,u)\delta^{im}\delta^{kl},\end{aligned} \tag{2.43}$$

where $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_3 + p_1)^2$.
The amplitude is invariant under simultaneous interchange of i with k and p_1 with

p_2; that means of t with u. Analogously one may interchange i with l or k with l. Doing so, one finally concludes that $B(s,t,u) = A(u,s,t)$ and $C(s,t,u) = A(t,u,s)$. So

$$\begin{aligned}\langle\pi^+(\vec{p}_1)\pi^-(\vec{p}_2)\pi^0(\vec{p}_3)|\bar{q}\lambda^3 q|\eta(\vec{p}_\eta)\rangle &= i(2\pi)^4\delta^4\left(p_{in}-p_{out}\right)A(s,t,u), \quad (2.44)\\ \langle\pi^0(\vec{p}_1)\pi^0(\vec{p}_2)\pi^0(\vec{p}_3)|\bar{q}\lambda^3 q|\eta(\vec{p}_\eta)\rangle &= i(2\pi)^4\delta^4\left(p_{in}-p_{out}\right)\bar{A}(s,t,u),\end{aligned}$$

with $\bar{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$.

2.7 Quark mass ratios $\frac{m_u}{m_s}$ and $\frac{m_d}{m_s}$ and elliptic constraint

The expressions for the mass squares of the pseudoscalars found in equation (2.25) imply the following first order relations for the quark masses:

$$\begin{aligned}2Bm_u &= \left(m^2_{K^+} - m^2_{K^0}\right)_{QCD} + m^2_{\pi^0},\\ 2Bm_d &= \left(m^2_{K^0} - m^2_{K^+}\right)_{QCD} + m^2_{\pi^0}, \quad (2.45)\\ 2Bm_s &= \left(m^2_{K^+} + m^2_{K^0}\right)_{QCD} - m^2_{\pi^0}.\end{aligned}$$

Using the Dashen theorem, we may express this in terms of the physical masses and find the leading order expressions

$$\begin{aligned}\frac{m_u}{m_s} &= \frac{m^2_{K^+} - m^2_{K^0} + 2m^2_{\pi^0} - m^2_{\pi^+}}{m^2_{K^+} + m^2_{K^0} - m^2_{\pi^+}} = 0.050 \quad \text{and}\\ \frac{m_d}{m_s} &= \frac{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}}{m^2_{K^+} + m^2_{K^0} - m^2_{\pi^+}} = 0.027\,. \quad (2.46)\end{aligned}$$

These are only estimates. To get results of higher accuracy one needs to take higher order contributions into account and to use other pieces of information as e.g. the ratio

$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2} \approx \frac{m_d^2 - m_u^2}{m_s^2},$$

which provides information in form of an elliptic constraint for the ratios m_s/m_d and m_u/m_d:

$$\frac{m_u^2}{m_d^2} + \frac{1}{Q^2}\frac{m_s^2}{m_d^2} = 1. \quad (2.47)$$

The dispersive analysis of the decay $\eta \to 3\pi$ in principle allows one to get quite precise knowledge of the numerical value of Q.
The following figure I took from [14]. It depicts the first quadrant of Leutwyler's ellipse (see equation (2.47)). The upper ellipse corresponds to $Q = 24.2$, the lower one to $Q = 21.5$. The lines restrict the allowed region for the ratios m_s/m_d and m_u/m_d further. The dashed ones one gets from $\eta\eta'$-mixing and the boundaries marked by the dotted lines are a consequence of baryon mass splittings, $\rho\omega$-mixing and $\Gamma(\psi' \to \psi\pi^0)/\Gamma(\psi' \to \psi\eta)$. I didn't look more closely at the arguments which lead to the dashed and dotted boundaries.

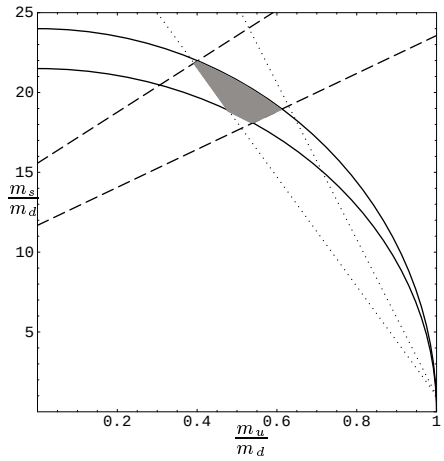


Figure 2.1: The shaded region is the region for the ratios m_s/m_d and m_u/m_d not excluded by the constraints mentioned in the text.

2.8 Elastic $\pi\pi$-scattering

2.8.1 Unitarity of the S matrix

The unitarity of the S matrix is crucial for the present work. Therefore I will give a short summary thereof. The scattering matrix S is defined by

$$S_{fi} = \langle f_{\text{out}} | i_{\text{in}} \rangle = \langle f \,{}_{\text{out}}^{\text{in}} | S | i \,{}_{\text{out}}^{\text{in}} \rangle. \tag{2.48}$$

Since the probability of having any final state $|f\rangle$ when starting with initial state $|i\rangle$ must be equal 1, we get the condition $\sum_f |\, \langle f|S|i\rangle \,|^2 = 1$, where we either use the *in*- or the *out*-states forming a complete set of orthonormal states.

$$\sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle = 1 \quad \Rightarrow \quad S^\dagger S = 1.$$

In an analogous way $SS^\dagger = 1$ is shown. So S has to be unitary. For the T-matrix, defined by $S = 1 + iT$ this implies

$$T - T^\dagger = iTT^\dagger. \tag{2.49}$$

The T matrix elements are written as

$$\langle f \,{}_{\text{out}}^{\text{in}} | T | i \,{}_{\text{out}}^{\text{in}} \rangle = i(2\pi)^4 \delta^4(p_f - p_i) T_{fi}.$$

Assuming TP invariance of the occurring states, $\langle f_{\text{out}}|i_{\text{in}}\rangle = \langle i_{\text{in}}|f_{\text{out}}\rangle^\star = \langle i_{\text{out}}|f_{\text{in}}\rangle$, we conclude $T_{fi} = T_{if}$. We write equation (2.49) in terms of T_{fi}:

$$i(2\pi)^4\delta^4(p_f - p_i)\left(T_{fi} - T^\star_{fi}\right) = i\left((2\pi)^4\right)^2 \sum_n \delta^4(p_f - p_n)\delta^4(p_n - p_i) T_{fn} T^\star_{in}.$$

For the elements T_{fi} the unitarity condition thus reads:

$$\text{Im}\, T_{fi} = \frac{(2\pi)^4}{2} \sum_n \delta^4(p_n - p_i) T_{fn} T^\star_{in}. \tag{2.50}$$

Note that the sum over n runs only over physical states $|n\rangle$.

2.8.2 Isospin decomposition

For further use we want to look more closely at elastic $\pi\pi$-scattering and at the isospin structure of the scattering amplitude.
To start with, consider a rank two tensor v transforming under rotations according to the representation $D^1 \otimes D^1$: $v^{ik} \to R^{il} R^{km} v^{lm}$. This tensor may be decomposed into three parts, v_0, v_1 and v_2, transforming according to D^0, D^1 and D^2, respectively. v_0 must be proportional to δ^{ik}, v_1 has to be antisymmetric and v_2 is required to be symmetric and traceless. These conditions may be satisfied by setting

$$\begin{aligned} v_0^{ik} &= \frac{\langle v \rangle}{3}\,\delta^{ik}\,, \\ v_1^{ik} &= \frac{1}{2}\left(v^{ik} - v^{ki}\right), \\ v_2^{ik} &= \frac{1}{2}\left(v^{ik} + v^{ki}\right) - \frac{\langle v \rangle}{3}\,\delta^{ik}\,. \end{aligned}$$

Define projection operators $P_I^{lm,ik}$, satisfying $v_I^{lm} = P_I^{lm,ik} v^{ik}$:

$$\begin{aligned} P_0^{lm,ik} &= \frac{1}{3}\delta^{ik}\delta^{lm}\,, \\ P_1^{lm,ik} &= \frac{1}{2}\left(\delta^{il}\delta^{km} - \delta^{im}\delta^{kl}\right), \\ P_2^{lm,ik} &= \frac{1}{2}\left(\delta^{il}\delta^{km} + \delta^{im}\delta^{kl}\right) - \frac{1}{3}\delta^{ik}\delta^{lm}. \end{aligned} \tag{2.51}$$

The so defined operators are real (hermitian) and fulfill $P_I^{lm,rs} P_J^{rs,ik} = \delta_{IJ} P_I^{lm,ik}$, as projection operators should. In the same way as for $\eta \to 3\pi$ we find that the transition matrix element for elastic $\pi\pi$-scattering is of the form

$$\begin{aligned} \left\langle \pi^l \pi^m | T | \pi^i \pi^k \right\rangle &= i\,(2\pi)^4 \delta(p_{in} - p_{out})\, T^{lm,ik} \\ &= i\,(2\pi)^4 \delta(p_{in} - p_{out}) (\delta^{lm}\delta^{ik} T_1 + \delta^{li}\delta^{mk} T_2 + \delta^{lk}\delta^{im} T_3)\,. \end{aligned}$$

Here and in the following the arguments of functions T_1, etc. have been omitted. Now apply the P_I's to two pion states $|\pi^i \pi^k\rangle$ and decompose them into total isospin I=0, 1 and 2 contributions $|\pi^i \pi^k\rangle_I = P_I^{ik,lm} |\pi^l \pi^m\rangle$. For $T^{lm,ik}$ this yields

$$\begin{aligned} i\,(2\pi)^4 \delta(p_{in} - p_{out})\, T^{lm,ik} &= \sum_{I,J} {}_J\left\langle \pi^l \pi^m | T | \pi^i \pi^k \right\rangle_I \\ &= \sum_I {}_I\left\langle \pi^l \pi^m | T | \pi^i \pi^k \right\rangle_I . \end{aligned} \tag{2.52}$$

We introduce the functions $A_I^{lm,ik}$ defined through:

$${}_I\left\langle \pi^l(p_3) \pi^m(p_4) | T | \pi^i(p_1) \pi^k(p_2) \right\rangle_I = i(2\pi)^4 \delta(p_{in} - p_{out}) A_I^{lm,ik}(p_1, p_2, p_3, p_4).$$

For $A_I^{lm,ik}$ we get the following expressions:

$$A_I^{lm,ik} = P_I^{lm,tu} P_I^{ik,rs} \left(\delta^{tu}\delta^{rs} T_1 + \delta^{tr}\delta^{us} T_2 + \delta^{ts}\delta^{ur} T_3\right) = P_I^{lm,ik} A_I.$$

where

$$\begin{aligned} A_0 &= 3\,T_1 + T_2 + T_3, \\ A_1 &= T_2 - T_3, \\ A_2 &= T_2 + T_3. \end{aligned} \tag{2.53}$$

For low energies, namely $s < 16m_\pi^2$, only two pion states contribute to $\sum_n$ in (2.50), because no additional meson pair may be created. Due to parity it isn't possible to create only one additional pion. In the low energy region the states $|n\rangle$ may thus be denoted by $|\pi^r(q_1)\pi^s(q_2)\rangle$. For the sum over intermediate states we then get

$$\sum_n = \frac{1}{2}\sum_{r,s=1}^{3}\int \frac{d^3q_1}{2q_1^0(2\pi^3)}\frac{d^3q_2}{2q_2^0(2\pi)^3}\,.$$

The factor $\frac{1}{2}$ is needed because $|\pi^r(p_1)\pi^s(p_2)\rangle = |\pi^s(p_2)\pi^r(p_1)\rangle$ and we would like to perform the sums over r and s independently of each other. The measure of the integrals is the same, Lorentz invariant measure as the one occurring in the Fourier decomposition of the field operator. We now make use of equation (2.50):

$$\begin{aligned}
\operatorname{Im} T^{lm,ik} &= \frac{(2\pi)^4}{4}\sum_{r,s}\int\frac{d^3q_1\,d^3q_2}{4\,q_1^0q_2^0\,(2\pi)^6}\,\delta^4(p_1+p_2-q_1-q_2)\,T^{lm,rs}T^{\star rs,ik}\\
&= \frac{1}{64\pi^2}\int\frac{d^3q_1\,d^3q_2}{q_1^0q_2^0}\sum_{rs}\delta^4(p_1+p_2-q_1-q_2)\times\\
&\quad\left(P_0^{lm,rs}A_0+P_1^{lm,rs}A_1+P_2^{lm,rs}A_2\right)\times\left(P_0^{rs,ik}A_0+P_1^{rs,ik}A_1+P_2^{rs,ik}A_2\right)^\star\\
&= \frac{1}{64\pi^2}\int\frac{d^3q_1\,d^3q_2}{q_1^0q_2^0}\delta(p_1+p_2-q_1-q_2)\times\\
&\qquad\qquad\left(P_0^{lm,ik}A_0A_0^\star+P_1^{lm,ik}A_1A_1^\star+P_2^{lm,ik}A_2A_2^\star\right)\\
&= P_0^{lm,ik}\operatorname{Im}A_0+P_1^{lm,ik}\operatorname{Im}A_1+P_2^{lm,ik}\operatorname{Im}A_2.
\end{aligned}\tag{2.54}$$

Restoring the arguments of the functions A_I this yields

$$\begin{aligned}
&\operatorname{Im}A_I(p_1,p_2,p_3,p_4) = \\
&\quad\frac{1}{64\pi^2}\int\frac{d^3q_1\,d^3q_2}{q_1^0q_2^0}\,\delta^4(p_1+p_2-q_1-q_2)\,A_I(q_1,q_2,p_3,p_4)\,A_I^\star(p_1,p_2,q_1,q_2).
\end{aligned}\tag{2.55}$$

To proceed further, we go over to the center of mass system: $\vec{p}_2=-\vec{p}_1$ and $\vec{p}_4=-\vec{p}_3$. The Dirac-δ^4 function can then be written as $\delta(\sqrt{s}-q_1^0-q_2^0)\delta^3(\vec{q}_1-\vec{q}_2)$ and thus cancels the integral over q_2. Beside that, being onshell, $\vec{q}_1=-\vec{q}_2$ implies $q_1^0=q_2^0$. To evaluate the remaining integral, note that $d^3q_1 = d\Omega\,dq\,q^2 = d\Omega\,dq_1^0\,q_1^0q$, where $q^2=|\vec{q}_1|^2=\left(q_1^{0\,2}-m_\pi^2\right)$. The functions A_I depend only on the total mass s and the scattering angle $\vartheta=\angle(\vec{p}_1,\vec{q}_1)$. We define the following additional two angles: $\vartheta_0=\angle(\vec{p}_1,\vec{p}_3)$ and $\vartheta'=\angle(\vec{q}_1,\vec{p}_3)$. Equation (2.55) now becomes:

$$\operatorname{Im}A_I(s,\vartheta_0)=\frac{1}{64\pi^2}\int d\Omega\,dq_1^0\,\frac{\sqrt{q_1^{0\,2}-m_\pi^2}}{q_1^0}\,\delta(\sqrt{s}-2q_1^0)\,A_I(s,\vartheta')\,A_I^\star(s,\vartheta).$$

or evaluating the integral over q_1^0:

$$\operatorname{Im}A_I(s,\vartheta_0)=\frac{1}{128\pi^2}\sqrt{\frac{s-4m_\pi^2}{s}}\int d\Omega\,A_I(s,\vartheta')\,A_I^\star(s,\vartheta).\tag{2.56}$$

For the A_I's we introduce the partial wave expansion:

$$A_I(s,\vartheta)=\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\vartheta)\,g_\ell^I(s),\tag{2.57}$$

where P_ℓ denote the Legendre polynomials, for which the following relation holds:

$$\int d\Omega\, P_\ell(\cos\vartheta) P_{\ell'}(\cos\vartheta') = \frac{4\pi}{2\ell+1}\,\delta_{\ell\ell'} P_\ell(\cos\vartheta_0)$$

So equation (2.56) finally reads:

$$\sum_\ell (2\ell+1) P_\ell(\cos\vartheta_0) \mathrm{Im}\, g_\ell^I(s) = \frac{\sqrt{1-4m_\pi^2/s}}{32\pi} \sum_\ell (2\ell+1) P_\ell(\cos\vartheta_0) g_\ell^I(s) g_\ell^{I\star}(s). \quad (2.58)$$

Since the P_ℓ are linearly independent, one gets

$$\mathrm{Im} g_\ell^I(s) = \frac{\sqrt{1-4m_\pi^2/s}}{32\pi}\, g_\ell^I(s)\, g_\ell^{I\star}(s),$$

which implies the following representation for the form factors g_ℓ^I:

$$g_\ell^I(s) = \frac{32\pi}{\sqrt{1-4m_\pi^2/s}}\, e^{i\delta_\ell^I(s)} \sin\delta_\ell^I(s). \quad (2.59)$$

Inserting this result into the expression for $A_I(s,\vartheta)$ we obtain

$$A_I(s,\vartheta) = \frac{32\pi}{\sqrt{1-4m_\pi^2/s}} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\vartheta) e^{i\delta_\ell^I(s)} \sin\delta_\ell^I(s).$$

Now look again at the symmetry properties of the transition amplitude. The matrix element

$$\langle \pi^l(\vec{p}_3)\pi^m(\vec{p}_4)|T|\pi^i(\vec{p}_1)\pi^k(\vec{p}_2)\rangle = i(2\pi)^4\delta(p_{out}-p_{in}) \left(P_0^{lm,ik} A_0(s,\vartheta) + P_1^{lm,ik} A_1(s,\vartheta) + P_2^{lm,ik} A_2(s,\vartheta) \right).$$

is invariant under a simultaneous interchange of l with m and $\vec{p}_3$ with $\vec{p}_4$, which means $\cos\vartheta \to -\cos\vartheta$. Because $P_\ell(-\cos\vartheta) = (-1)^\ell P_\ell(\cos\vartheta)$ and $P_I^{ml,ik} = (-1)^I P_I^{lm,ik}$ only those contributions can survive with both, I and ℓ, being odd or both being even. If we restrict the sum over partial waves to S- and P-waves, we are in the position to write down an approximation for $T^{lm,ik}$ which will be made use of in the next section:

$$T^{lm,ik}(s,\cos\vartheta) = \frac{32\pi}{\sqrt{1-4m_\pi^2/s}} \Big\{ P_0^{lm,ik} e^{i\delta_0(s)} \sin\delta_0(s) + 3\cos\vartheta P_1^{lm,ik} e^{i\delta_1(s)} \sin\delta_1(s) + P_2^{lm,ik} e^{i\delta_2(s)} \sin\delta_2(s) \Big\}. \quad (2.60)$$

$\delta_0 = \delta_0^0$, $\delta_1 = \delta_1^1$ and $\delta_2 = \delta_0^2$ are the so called phase shifts.
At the present approximation the functions A_I are given by

$$\begin{aligned} A_0(s,\vartheta) &= \frac{32\pi}{\sqrt{1-4m_\pi^2/s}} \sin\delta_0(s) e^{i\delta_0(s)} \\ A_1(s,\vartheta) &= \frac{96\pi\cos\vartheta}{\sqrt{1-4m_\pi^2/s}} \sin\delta_1(s) e^{i\delta_1(s)} \\ A_2(s,\vartheta) &= \frac{32\pi}{\sqrt{1-4m_\pi^2/s}} \sin\delta_2(s) e^{i\delta_2(s)}. \end{aligned} \quad (2.61)$$

2.8.3 Elastic $\pi\pi$-scattering to one loop in χPT

Although we are dealing with the decay $\eta \to 3\pi$ on these pages we want to consider elastic $\pi\pi$-scattering to one loop, for the following reason: Starting with the matrix element for the η-decay, we may transform the isospin quantum number of the isospin breaking operator $\frac{1}{2}(m_u - m_d)\langle\pi^i\pi^k\pi^l|\bar{q}\lambda^3 q|\eta\rangle$ to the fourth particle – we get a π^3 instead of the isosinglet η. For kinematic reasons we have to push one π from the out state to the in state. We may therefore consider the amplitude $\pi^i(p_1)\pi^k(p_2) \to \pi^l(p_3)\pi^m(p_4)$ instead of the one for $\eta \to 3\pi$. For $m = 3$ the two corresponding matrix elements will be of the same isospin structure. In the case of elastic $\pi\pi$-scattering the Mandelstam variables become

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - p_3)^2 \quad \text{and} \quad u = (p_1 - p_4)^2\,, \quad s + t + u = 4m_\pi^2\,.$$

The calculation will be presented rather detailed and can, maybe, serve as a sample. Another advantage of considering elastic $\pi\pi$-scattering instead of η-decay is, that it is sufficient to use an $SU(2) \times SU(2)$ effective chiral Lagrangian, which solely involves pion fields – which all have the same mass within QCD. The small difference induced by $\eta\pi^0$-mixing shows up only in the $SU(3) \times SU(3)$ description. Considering the η-decay on the other hand would require an $SU(3) \times SU(3)$ Lagrangian that gives rise to more diagrams and involves particles of different masses. A further simplification is to consider the isospin symmetric limit with $m_u = m_d = \hat{m}$. This way the whole quark mass dependence may be described through the square of the bare pion mass $m^2 \doteq 2B\hat{m}$.

In $d = 4$ dimensions the loop expansion is equivalent to an expansion in powers of the momenta or inverse powers of F^2, respectively. To get the one-loop amplitude we therefore need to take into account all diagrams generated by $\mathcal{L}_{\text{eff}}^{(2)}$ up to and including $\mathcal{O}(F^{-4})$ and all tree graphs associated with $\mathcal{L}_{\text{eff}}^{(4)}$. To get rid of the infinities caused by the one-loop integrals we will use dimensional regularization which is very appropriate to χPT.
It is favourable to perform the calculation in the octet basis – as already mentioned, the $\pi\pi$-scattering amplitude is of the form

$$\begin{aligned}\left\langle \pi^i(p_1)\pi^k(p_2) \mid T \mid \pi^l(p_3)\pi^m(p_4)\right\rangle = i(2\pi)^4\delta^4\left(p_1 + p_2 - p_3 - p_4\right) \times \\ \left(\delta^{ik}\delta^{lm}A(s,t,u) + \delta^{il}\delta^{km}A(t,u,s) + \delta^{im}\delta^{kl}A(u,s,t)\right)\,.\end{aligned}$$

When extracting contributions to the four point functions it is therefore sufficient to consider diagrams with $i = k \neq l = m$. In the following calculation we will stick to $i = k = 1$ and $l = m = 2$. As a check of the calculation you might evaluate in addition the two diagrams with $(k \leftrightarrow l)$ and $(k \leftrightarrow m)$ as well as the diagram belonging to $(i = k = l = m)$. The contribution from the latter has to be equal to the sum of the first three. This holds separately for every diagram, irrespective of the topology .
The $\mathcal{O}(p^2)$ effective Lagrangian is given by:

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{F^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger + m^2(U + U^\dagger)\right\rangle\,, \quad U = \exp(i/F\phi).$$

Since we are within the $SU(2) \times SU(2)$ framework, we have

$$\phi = \begin{pmatrix} \pi^3 & \pi^1 - i\pi^2 \\ \pi^1 + i\pi^2 & -\pi^3 \end{pmatrix}.$$

The $\mathcal{O}(p^4)$ part of the effective Lagrangian is

$$\mathcal{L}_{\text{eff}}^{(4)} = \frac{l_1}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger\right\rangle^2 + \frac{l_2}{4}\left\langle \partial_\mu U \partial_\nu U^\dagger\right\rangle\left\langle \partial^\mu U \partial^\nu U^\dagger\right\rangle + \frac{l_3}{4}\left\langle \chi U\right\rangle^2 + \frac{l_4}{4}\left\langle \partial_\mu \chi^\dagger \partial^\mu U\right\rangle. \tag{2.62}$$

Performing a partial integration, the term $\langle \partial_\mu \chi^\dagger \partial^\mu U \rangle$ may be replaced by

$$-\langle \chi^\dagger \Box U \rangle = \langle \chi^\dagger U \rangle \langle \partial_\mu U \partial^\mu U^\dagger \rangle - \langle \chi \chi^\dagger \rangle + \langle \chi^\dagger U \rangle^2 .$$

All Feynman diagrams occurring in the present calculation are depicted in appendix (A.3).
Retaining only terms $\sim (\pi^1)^2 (\pi^2)^2$, the interaction Lagrangian responsible for the treelevel contribution from $\mathcal{L}_{\text{eff}}^{(2)}$ becomes

$$\mathcal{L}_{\text{tree}}^{(2)} = \frac{m^2}{12F^2}\left(\pi^{1\,2}\pi^{2\,2}\right) + \frac{1}{6F^2}\Big(2\partial_\mu\pi^1\pi^1\partial^\mu\pi^2\pi^2 - \partial_\mu\pi^1\partial^\mu\pi^1\pi^{2\,2} - \pi^{1\,2}\partial_\mu\pi^2\partial^\mu\pi^2\Big).$$

The non-derivative term gives rise to

$$2 \times 2 \times \frac{m^2}{12F^2} = \frac{m^2}{3F^2} ,$$

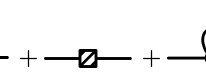
where 2×2 is the corresponding combinatorical factor.
From the energy dependent part of $\mathcal{L}_1$ we get

$$\frac{1}{6F^2}\big(2 \times (p_1 + p_2)(p_3 + p_4) + 2 \times 2 \times p_1 p_2 + 2 \times 2 \times p_3 p_4\big) = \frac{s - \frac{4}{3}m_\pi^2}{F^2} .$$

Summing up the two parts yields


$$= \tilde{T}(s) = \frac{s - \frac{4}{3}m_\pi^2 + \frac{1}{3}m^2}{F^2} . \tag{2.63}$$

In the pure treelevel calculation where the $\mathcal{O}(F^{-4})$ difference $m_\pi^2 - m^2$ may be neglected, equation (2.63) represents Weinberg's formula for the elastic $\pi\pi$-scattering amplitude. For $\pi^i\pi^i \to \pi^i\pi^i$ the treelevel approximation of the amplitude is energy independent.
The terms in $\mathcal{L}_{\text{eff}}^{(4)}$ proportional to l_3 and l_4 yield a contribution to the kinetic part of the Lagrangian and thus contribute to the propagator. Including the tadpole terms, the inverse propagator is then given by

$$\Delta'\left(p^2\right)^{-1} = \left(\text{  }\right)^{-1} = m^2 + \Sigma\left(p^2\right) - p^2$$
$$= m^2\left(1 + \frac{2m^2}{F^2}\left(l_3 + l_4\right) + \frac{i\Delta(0)}{6F^2}\right) - p^2\left(1 + \frac{2m^2}{F^2}\,l_4 + \frac{2i\Delta(0)}{3F^2}\right), \tag{2.64}$$

where $\Sigma\left(p^2\right)$ denotes the self-energy. The renormalized field ϕ^r and the physical mass m_π are connected to the bare field and mass through

$$\phi = \sqrt{Z}\phi_r , \quad \text{and} \quad m_\pi^2 = m^2 + \delta m^2 .$$

The physical mass is determined by the requirement that $\Delta'\left(p^2\right)$ has a pole at $p^2 = m_\pi^2$ and Z is given by the residue of this pole:

$$\delta m^2 = \Sigma\left(m_\pi^2\right) \quad \text{and} \quad Z = \frac{1}{1 - \Sigma'\left(m_\pi^2\right)} .$$

From equation (2.64) we thus get

$$\delta m^2 = \frac{2m^4}{F^2}\,l_3 - \frac{m^2}{2F^2}\,i\Delta(0) , \quad \text{and} \quad Z = \left(1 - \frac{2m^2}{F^2}\,l_4 - \frac{2}{3F^2}i\Delta(0)\right). \tag{2.65}$$

The singularity in δm^2 arising with $\Delta(0)$ can be absorbed in l_3 by an appropriate renormalization:

$$l_3 = l_3^r + \gamma_3 \lambda\,, \quad \gamma_3 = -\frac{1}{2}\,, \tag{2.66}$$

where λ is defined in equation (A.4).
$\Delta(0)$ contains a term proportional to $\ln \frac{m^2}{\mu^2}$ referred to as a chiral logarithm (χlog). This χlog can be hidden defining

$$\bar{l}_i = \frac{32\pi^2 l_i^r}{\gamma_i} - \ln\frac{m^2}{\mu^2}\,. \tag{2.67}$$

The final result for m_π^2 then reads:

$$m_\pi^2 = m^2\left(1 - \frac{1}{32\pi^2}\frac{m^2}{F^2}\,\bar{l}_3\right). \tag{2.68}$$

The external line insertions have two effects. Firstly they replace the the bare mass in the external line factors by the physical one. Secondly the truncated one particle irreducible four point function receives an additional factor Z^2. At one-loop order, this is only relevant for the treelevel contribution $\widetilde{T}(s)$. All other corrections due to field renormalization are of $\mathcal{O}\left(p^6\right)$.
Next we consider the treelevel graphs generated by $\mathcal{L}_{\text{eff}}^{(4)}$. Their contribution to the amplitude is

$$= T_1(s) + T_2(s,t,u) + T_3 + T_4(s) =$$
$$\frac{2l_1}{F^4}\left(s-2m^2\right)^2 + \frac{l_2}{2F^4}\left(s^2+(t-u)^2\right) + \frac{8m^4}{3F^4}\,l_3 + \frac{4m^2}{F^2}\,l_4\cdot T(s)\,. \tag{2.69}$$

The term proportional to l_3 together with $\widetilde{T}(s)$ (equation (2.63)), taking into account the field renormalization, yields, up to $\mathcal{O}(F^{-4})$,

$$Z^2\,\widetilde{T}(s) + T_3 = Z^2\cdot\frac{s-m^2}{F^2} + \frac{2m^2}{3F^4}\,i\Delta(0) = T(s) + R_1(s) + D_1(s), \tag{2.70}$$

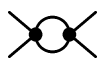
where

$$T(s) = \frac{s-m^2}{F^2} \qquad R_1(s) = -\frac{4m^2}{F^2}\,l_4\cdot T(s) \quad \text{and} \quad D_1(s) = \left(-\frac{4s}{3}+2m^2\right)\frac{i\Delta(0)}{F^4}\,.$$

The term $R_1(s)$ cancels the treelevel contribution $T_4(s)$ and the scattering amplitude gets independent of the low energy constant l_4 as it should in view of (2.62).
Now we turn to the loops. The Feynman rule associated with the vertex  , keeping only terms $\mathcal{O}(F^{-2})$, reads:

$$\frac{i}{F^2}\Big(\delta^{ik}\delta^{lm}\left(s-m^2\right) + \delta^{il}\delta^{km}\left(t-m^2\right) + \delta^{im}\delta^{kl}\left(u-m^2\right)\Big)\,.$$

From this we get the s-channel one-loop contribution from $\mathcal{L}_{\text{eff}}^{(2)}$:



$$= \frac{-i}{2F^4}\int\frac{d^dk}{(2\pi)^d}\sum_{r,s=1}^{3}\left(m^2-k^2\right)^{-1}\left(m^2-(p_1+p_2-k)^2\right)^{-1}\times$$
$$\Big\{\delta^{ik}\delta^{rs}\left(s-m^2\right) + \delta^{ir}\delta^{ks}\left((p_1-k)^2-m^2\right) + \delta^{is}\delta^{kr}\left((p_2-k)^2-m^2\right)\Big\}\times$$
$$\Big\{\delta^{rs}\delta^{lm}\left(s-m^2\right) + \delta^{rl}\delta^{sm}\left((p_3-k)^2-m^2\right) + \delta^{rm}\delta^{sl}\left((p_4-k)^2-m^2\right)\Big\},$$

where we have omitted the $i\epsilon$-prescription in the internal line factors. Compared to treelevel, we have one additional vertex and two internal lines, giving rise to a factor i and i^{-2}, respectively. This amounts to an overall factor of $-i$. The factor $\frac{1}{2}$ arises, because the symmetry factor associated with the internal lines is only 2 and not 4 as obtained by multiplying the expressions for the two vertices.
The formulas occurring at intermediate steps of the evaluation of the above forest would be too long to be written down here – I will skip these. Summing over r and s, we get contributions which all can be expressed in terms of the integrals presented in appendix (A.2). This means that the analytic expression belonging to  can be expressed in terms of the one-loop function $J(s)$.
Not only the s-channel provides terms proportional to $\delta^{ik}\delta^{lm}$, but also the t- and u-channel. However, replacing s by t (u) in the s-channel contribution proportional to $\delta^{il}\delta^{km}$ $(\delta^{im}\delta^{kl})$ is equal the $\delta^{ik}\delta^{lm}$ contribution from the t-channel (u-channel). Writing $J(s)$ as $\bar{J}(s) + J(0)$ we get the contribution B(s,t,u) to A(s,t,u), which includes all terms proportional to $\bar{J}$:

$$\begin{aligned} B(s,t,u) = \frac{1}{6F^4}\Big\{ & 3\left(s^2-m^4\right)\bar{J}(s) + \\ & \left(t(t-u) - 2m^2t + 4m^2u - 2m^4\right)\bar{J}(t) + \\ & \left(u(u-t) - 2m^2u + 4m^2t - 2m^4\right)\bar{J}(u)\Big\}\,. \end{aligned} \tag{2.71}$$

The remaining terms arising from $\left(\text{  } + crossed\right)$ proportional to $\delta^{ik}\delta^{lm}$ we collect in

$$\begin{aligned} \widetilde{C}(s,t,u) = & \frac{1}{6F^4}\left\{2\left(s-2m^2\right)^2 + \left(s^2+(t-u)^2\right) - 3m^4 + 18sm^2\right\}J(0) + \\ & \frac{1}{288\pi^2F^4}\left\{\frac{1}{2}\left(s^2+(t-u)^2\right) - 2\left(s-2m^2\right)^2 + 36m^4 + 18sm^2\right\}. \end{aligned} \tag{2.72}$$

The singular terms proportional to $\left(s-2m^2\right)^2$ and $s^2+(t-u)^2$ amount to a renormalization of l_1 and l_2, respectively:

$$l_1 = l_1^r + \gamma_1\lambda\,, \quad l_2 = l_2^r + \gamma_2\lambda\,, \quad \text{where} \quad \gamma_1 = \frac{1}{3} \quad \text{and} \quad \gamma_2 = \frac{2}{3}\,. \tag{2.73}$$

The corresponding χlogs are again hidden according to (2.67). Adding $\widetilde{C}(s,t,u)$ and the treelevel amplitudes $T_1(s)$ and $T_2(s,t,u)$ up to $C(s,t,u) + D_2(s)$ then yields

$$\begin{aligned} C(s,t,u) = \frac{1}{96\pi^2F^4}\Big\{ & 2\left(\bar{l}_1 - \frac{4}{3}\right)\left(s-2m^2\right)^2 + \\ & \left(\bar{l}_2 - \frac{5}{6}\right)\left(s^2+(t-u)^2\right) + 15m^4 - 12sm^2\Big\}, \end{aligned} \tag{2.74}$$

and

$$D_2(s) = \frac{1}{F^4}\left(3s - \frac{m^2}{2}\right) i\Delta(0).$$

At the one-loop level the graph  also needs to be taken into account. The relevant part of the Lagrangian is

$$\mathcal{L}_2 = \frac{1}{720F^4}\Big(-\frac{m^2}{2}\left\langle\phi^6\right\rangle + \left\langle\partial_\mu\phi\,\partial^\mu\phi\,\phi^4\right\rangle - 4\left\langle\partial_\mu\phi\,\phi\,\partial^\mu\phi\,\phi^3\right\rangle + 3\left\langle\partial_\mu\phi\,\phi^2\,\partial^\mu\phi\,\phi^2\right\rangle\Big)\,.$$

Again it is sufficient to consider only the terms relevant for $\pi^1\pi^1 \to \pi^2\pi^2$. The contribution from this graph cancels the terms proportional to $\Delta(0)$ that have not yet been removed by the renormalization procedure.

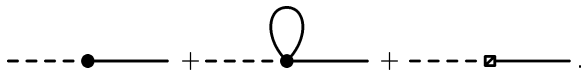
The total one-loop elastic $\pi\pi$-scattering amplitude, expressed in terms of the bare mass and pion decay constant is then given by

$$A(s,t,u) = T(s) + B(s,t,u) + C(s,t,u). \tag{2.75}$$

Now we work out the pion decay constant F_π to one-loop order. $\mathcal{L}_{\text{eff}}^{(4)}$ gives rise to additional contributions to the axial vector current, which becomes

$$\begin{aligned} A_\mu^a = {} & \frac{iF^2}{4}\left\langle \lambda^a \left\{\partial_\mu U, U^\dagger\right\}\right\rangle + \frac{i\,l_1}{2}\left\langle \lambda^a \left\{\partial_\mu U, U^\dagger\right\}\right\rangle \left\langle \partial_\nu U \partial^\nu U^\dagger\right\rangle + \\ & \frac{i\,l_2}{2}\left\langle \lambda^a \left\{\partial^\nu U, U^\dagger\right\}\right\rangle \left\langle \partial_\mu U \partial_\nu U^\dagger\right\rangle + \frac{i\,l_4}{4}\left\langle \chi^\dagger U\right\rangle \left\langle \lambda^a \left\{\partial_\mu U, U^\dagger\right\}\right\rangle . \end{aligned} \tag{2.76}$$

At one-loop order, $\left\langle 0|A_\mu^a|\pi^b(p)\right\rangle = ip_\mu \delta^{ab} F_\pi$ picks up no contribution from l_1 or l_2, but l_4 and the one-loop term associated with $\mathcal{L}_{\text{eff}}^{(2)}$ do generate a contribution. Further the leading order term receives a factor of $\sqrt{Z}$. Graphically we may describe this by the following sum of diagrams:



The corresponding analytic expression is

$$\left\langle 0|A_\mu^a|\pi^b(p)\right\rangle = ip_\mu \delta^{ab} F \left(1 + \frac{m^2}{F^2}\, l_4 + \frac{i\Delta(0)}{F^2}\right) = ip_\mu \delta^{ab} F_\pi .$$

We finally read off the necessary renormalization of l_4:

$$l_4 = l_4^r + \gamma_4 \lambda\,, \quad \gamma_4 = 2.$$

At $\mathcal{O}\left(p^4\right)$ the pion decay constant F_π is thus given by

$$F_\pi = F\left(1 + \frac{1}{16\pi^2}\frac{m^2}{F^2}\,\bar{l}_4\right). \tag{2.77}$$

The amplitude $A(s,t,u)$ may be written in terms of three functions depending on a single variable:

$$A(s,t,u) = A_0(s) + (s-u)A_1(t) + (s-t)A_1(u) + A_2(t) + A_2(u) - \frac{2}{3}A_3(s)\,, \tag{2.78}$$

with

$$\begin{aligned} A_0(s) = {} & T(s) + \frac{m^4 + 4sT(s)}{6F^4}\,\bar{J}(s) + \\ & \frac{2\left(s-2m^2\right)^2\left(\bar{l}_1 - \frac{4}{3}\right) + \left(2s^2 - \frac{4}{3}sm^2\right)\left(\bar{l}_2 - \frac{5}{6}\right) + 15m^4 - 12m^2 s}{96\pi^2 F^4}\,, \\ A_1(s) = {} & \frac{s-4m^2}{12F^4}\,\bar{J}(s) + \frac{s\left(\bar{l}_2 - \frac{5}{6}\right)}{192\pi^2 F^4}\,, \\ A_2(s) = {} & \frac{\left(s-2m^2\right)^2}{4F^4}\,\bar{J}(s) + \frac{s\left(3s-4m^2\right)\left(\bar{l}_2 - \frac{5}{6}\right)}{192\pi^2 F^4}\,. \end{aligned} \tag{2.79}$$

2.9 The χPT one-loop result for $\eta \to 3\pi$

Since the χPT one-loop result for $\eta \to 3\pi$ will be used to determine the subtraction constants in the dispersive analysis of the η-decay, I will quote it in this section. I have merely gathered the formulas presented in ref. [6] and [7].
It is convenient to extract a normalization factor:

$$\begin{aligned} A(s,t,u) &= -\frac{1}{Q^2}\frac{m_K^2(m_K^2-m_\pi^2)}{3\sqrt{3}m_\pi^2 F_\pi^2}M(s,t,u)\,, \\ \frac{1}{Q^2} &= \frac{m_d^2-m_u^2}{m_s^2-\hat{m}^2}\,. \end{aligned} \tag{2.80}$$

The normalization factor is chosen such that the current algebra result,

$$\overline{M}(s,t,u) = T(s) = 1 + 3\frac{s-s_0}{m_\eta^2-m_\pi^2}\,,$$

is equal to 1 at the center of the Dalitz plot ($s = s_0$).
Like the $\pi\pi$-amplitude, $\overline{M}(s,t,u)$ may be represented by three functions depending on a single variable:

$$\overline{M}(s,t,u) = \overline{M}_0(s)+(s-u)\overline{M}_1(t)+(s-t)\overline{M}_1(u)+\overline{M}_2(t)+\overline{M}_2(u)-\frac{2}{3}\overline{M}_2(s). \tag{2.81}$$

The index labels the isospin quantum number of the corresponding partial wave. The functions $M_I(s)$ are given by:

$$\begin{aligned} \overline{M}_0(s) &= T(s) + \frac{1}{3}\Delta_0(s)\,(3+2\,T(s)) + \Delta_3(s) + \frac{1}{3}\Delta_2(s)\,(3-T(s)) + \\ &\qquad V(s) + \frac{2}{3}\Delta_{GMO}T(s) + \frac{8}{3}\frac{m_\pi^2}{m_\eta^2-m_\pi^2}\,(\Delta_F T(s) - \Delta_{GMO})\,, \\ \overline{M}_1(s) &= \frac{3}{2(m_\eta^2-m_\pi^2)}\left(\Delta_1(s) - \frac{8}{3F_\pi^2}sL_3\right), \\ \overline{M}_2(s) &= \frac{1}{2}\Delta_2(s)\,(3-T(s))\,. \end{aligned} \tag{2.82}$$

The coupling constant L_3 is $L_3 = (-3.5 \pm 1.1)\cdot 10^{-3}$ and the quantities Δ_F and Δ_{GMO} are given by

$$\Delta_F \equiv \frac{F_K}{F_\pi} - 1 = 0.22\,, \qquad \Delta_{GMO} \equiv \frac{4m_K^2 - 3m_\eta^2 - m_\pi^2}{m_\eta^2 - m_\pi^2} = 0.21\,.$$

For further use we introduce the abbreviations $\Delta_{PQ} = m_P^2 - m_Q^2$ and $\Sigma_{PQ} = m_P^2 + m_Q^2$, where P and Q denote one of the particles π, η or K. The functions

$\Delta_i(s)$ are expressed in terms of the renormalized loop integrals $J^r_{PQ}(s)$ and $M^r_{PP}(s)$:

$$\begin{aligned}
\Delta_0(s) &= \frac{1}{2F_\pi^2}\left(2s - m_\pi^2\right) J^r_{\pi\pi}(s)\,,\\
\Delta_1(s) &= \frac{2s}{F_\pi^2}\left(M^r_{\pi\pi}(s) + \frac{1}{2}M^r_{KK}(s)\right),\\
\Delta_2(s) &= -\frac{1}{2F_\pi^2}\left(s - 2m_\pi^2\right) J^r_{\pi\pi}(s) + \frac{1}{4F_\pi^2}\left(3s - 4m_K^2\right) J^r_{KK}(s) + \frac{m_\pi^2}{3F_\pi^2} J^r_{\pi\eta}(s)\,,\\
\Delta_3(s) &= -\frac{1}{6F_\pi^2\Delta_{\eta\pi}}\left(s - 2m_\pi^2\right)\left(3s - 4m_K^2\right) J^r_{\pi\pi}(s)\\
&\quad - \frac{s\left(3s - 4m_\pi^2\right)}{4F_\pi^2\Delta_{\eta\pi}} J^r_{KK}(s) + \frac{m_\pi^2}{3F_\pi^2\Delta_{\eta\pi}}\left(3s - 4m_\pi^2\right) J^r_{\pi\eta}(s)\\
&\quad - \frac{m_\pi^2}{2F_\pi^2} J^r_{\eta\eta}(s) - \frac{3s}{8F_\pi^2}\frac{3s - 4m_K^2}{s - 4m_K^2}\left(J^r_{KK}(s) - J^r_{KK}(0) - \frac{1}{8\pi^2}\right).
\end{aligned} \tag{2.83}$$

J_{PQ} and M_{PP} are of the same nature as the loop integral $J(s)$ encountered in the elastic $\pi\pi$-scattering calculation – they just get more complicated because of the different masses occurring. J^r_{PQ} and M^r_{PP} take the form:

$$\begin{aligned}
J^r_{PQ}(s) &= \bar{J}_{PQ}(s) - 2k_{PQ}\,,\\
M^r_{PP}(s) &= \frac{1}{12s}\left(s - 4m_P^2\right)\bar{J}_{PP}(s) - \frac{1}{6}k_{PP} + \frac{1}{288\pi^2}\,,
\end{aligned} \tag{2.84}$$

with

$$\begin{aligned}
k_{PQ} &= \frac{1}{32\pi^2}\frac{m_P^2\ln\left(m_P^2/\mu^2\right) - m_Q^2\ln\left(m_Q^2/\mu^2\right)}{\Delta_{PQ}}\,,\\
\bar{J}_{PQ}(s) &= \frac{1}{32\pi^2}\left\{2 + \left(\frac{\Delta_{PQ}}{s} - \frac{\Sigma_{PQ}}{\Delta_{PQ}}\right)\ln\frac{m_Q^2}{m_P^2} - \frac{\nu}{s}\ln\frac{(s+\nu)^2 - \Delta^2}{(s-\nu)^2 - \Delta^2}\right\},\\
\nu^2 &= \Big\{s - (m_P + m_Q)^2\Big\}\Big\{s - (m_P - m_Q)^2\Big\}.
\end{aligned} \tag{2.85}$$

In the case where $m_P = m_Q = m$ these quantities take simpler forms:

$$\begin{aligned}
k_{PP} &= \frac{1}{32\pi^2}\left(\ln\frac{m^2}{\mu^2} + 1\right),\\
\bar{J}_{PP}(s) &= \frac{1}{16\pi^2}\left(\sigma\ln\frac{\sigma - 1}{\sigma + 1} + 2\right),\\
\sigma &= \sqrt{\frac{s - 4m^2}{s}}\,.
\end{aligned}$$

The function $V(s)$ is given by

$$\begin{aligned}
V(s) &= T(s)\left(a_1 + 3a_2\Delta_{\eta\pi} + a_3\left(9m_\eta^2 - m_\pi^2\right)\right) + a_4\,,\\
a_1 &= \frac{2}{3\Delta_{\eta\pi}}m_\pi^2\left(-\mu_\pi - 2\mu_K + 3\mu_\eta\right),\\
a_2 &= \frac{2}{3\Delta_{\eta\pi}^2}\left(-m_\pi^2\mu_\pi + 4m_K^2\mu_K - 3m_\eta^2\mu_\eta\right),\\
a_3 &= \frac{1}{128\pi^2F_\pi^2}\left(\ln\frac{m_K^2}{\mu^2} + 1\right),\\
a_4 &= -8m_\pi^2a_2 - 12m_K^2a_3 + \frac{3m_\pi^2}{32\pi^2F_\pi^2}\left(1 - \frac{m_\pi^2}{m_K^2 - m_\pi^2}\ln\frac{m_K^2}{m_\pi^2}\right).
\end{aligned} \tag{2.86}$$

The renormalization scale dependent quantities μ_P are the χlogs

$$\mu_P = \frac{1}{32\pi^2 F_\pi^2} m_P^2 \ln \frac{m_P^2}{\mu^2}. \tag{2.87}$$

Except for the terms proportional to Δ_F, Δ_{GMO} and L_3 as well as for the treelevel contribution, the one loop formula is accurate only to leading order in the quark mass expansion. At leading order the meson masses satisfy the Gell-Mann-Okubo relation (equation (2.29)). The final result is then independent of the renormalization scale μ. Putting all pieces together, we get the following plot for the amplitude $\overline{M}(s,t,u)$ along the line $s = u$:

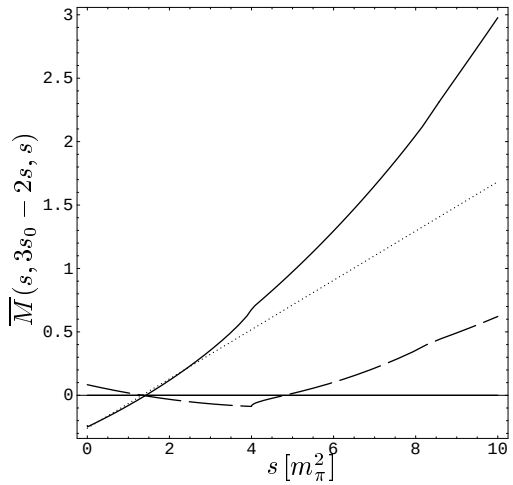
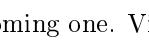


Figure 2.2: One-loop χPT result of $\overline{M}(s,t,u)$ along the line $s = u$. The full line is the real part, the dashed one represents the imaginary part and the dotted line depicts the current algebra prediction $T(s)$.

2.10 Implications of unitarity

Consider the transition amplitude $A_n \doteq A_{\eta\to n}$, where the final state is a three-pion state denoted by $|n_{\rm out}\rangle$. A_n is given by $\langle n_{\rm out}|\mathcal{L}|\eta\rangle$. In view of $\langle n_{\rm out}|\mathcal{L}|\eta\rangle = \langle n_{\rm in}|\mathcal{L}|\eta\rangle^\star$ we get for the imaginary part ofA_n:

$$\begin{aligned}
\mathrm{Im}\, A_n &= \frac{1}{2i}\Big\{\langle n_{\rm out}|\mathcal{L}|\eta\rangle - \langle n_{\rm in}|\mathcal{L}|\eta\rangle\Big\} \\
&= \frac{1}{2i}\sum_{n'}\Big\{\langle n_{\rm out}|n'_{\rm out}\rangle\langle n'_{\rm out}|\mathcal{L}|\eta\rangle - \langle n_{\rm in}|n'_{\rm out}\rangle\langle n'_{\rm out}|\mathcal{L}|\eta\rangle\Big\} \\
&= \frac{1}{2i}\sum_{n'}\Big\{\delta_{nn'} - \langle n_{\rm in}|n'_{\rm out}\rangle\Big\}A_{n'} \\
&= \frac{1}{2i}\sum_{n'}\Big\{\delta_{nn'} - \langle n'_{\rm out}|n_{\rm in}\rangle^\star\Big\}A_{n'} \\
&= \frac{1}{2i}\sum_{n'}\Big\{\delta_{nn'} - \big(\delta_{nn'} + i(2\pi)^4\delta^4(p_n - p_{n'})T_{n'n}\big)^\star\Big\}A_{n'} \\
&= \frac{1}{2}\sum_{n'}\Big\{(2\pi)^4\delta^4(p_n - p_{n'})T^\star_{n'n}\Big\}A_{n'}.
\end{aligned} \tag{2.88}$$

For the decay $\eta \to 3\pi$ the sum over the intermediate states n' runs only over states containing three pions, as long as we are in the physical region. This is due to energy-momentum conservation. Especially there will be no other η involved than the incoming one. Virtual transitions, such as  contribute to the self-energy of the η and are incorporated in the physical mass m_η. The above unitarity condition thus reduces to a linear constraint of the amplitude A_n. We are therefore allowed to extract a normalization factor:

$$\begin{aligned} A(s,t,u) &= -\frac{1}{Q^2}\frac{m_K^2(m_K^2-m_\pi^2)}{3\sqrt{3}m_\pi^2 F_\pi^2}M(s,t,u), \\ \frac{1}{Q^2} &= \frac{m_d^2-m_u^2}{m_s^2-\hat{m}^2}\,. \end{aligned} \tag{2.89}$$

Chiral perturbation theory shows that, up to and including two loops, the low energy amplitude is dominated by two-body collision finalstate interactions, described by elastic $\pi\pi$-scattering, which, in this region, only occurs in the S- and P-waves. In the following we assume that the amplitude can by described in the same way as the one-loop result from χPT:

$$M(s,t,u) = M_0(s)+(s-u)M_1(t)+(s-t)M_1(u)+M_2(t)+M_2(u)-\frac{2}{3}M_2(s). \tag{2.90}$$

An analogous representation is valid at two loop order of χPT ([17]).
We return to the general case $\eta \to \pi^i(\vec{p}_1)\pi^k(\vec{p}_2)\pi^l(\vec{p}_3)$ and use

$$M^{ikl}(s,t,u) = \delta^{ik}\delta^{l3}M(s,t,u) + \delta^{kl}\delta^{i3}M(t,u,s) + \delta^{li}\delta^{k3}M(u,s,t). \tag{2.91}$$

The unitarity condition in terms of M now reads:

$$\operatorname{Im} M^{ikl}(s,t,u)\Big|_d = \frac{1}{2}\sum_{n'}(2\pi)^4\delta(p_n-p_{n'})T^\star_{n'n}M_{n'}\,.$$

M^{ikl} denotes the amplitude for $\eta \to \pi^i\pi^k\pi^l$, M_n and $T_{n'n}$ the one for $\eta \to n$ and $n \to n'$, respectively. Restricting the final state interaction to two-body collisions, $T_{n'n}$ is given by the functions $T^{lm,ik}$. The sum over the intermediate states n' then yields the same sum as already met in the partial wave expansion:

$$\sum_{n'} = \frac{1}{2}\sum_{a,b=1}^{3}\int\frac{d^3p_a}{2p_a^0(2\pi^3)}\frac{d^3p_b}{2p_b^0(2\pi)^3}\,.$$

We get

$$\begin{aligned} \operatorname{Im} M^{ikl}(s,t,u)\Big|_d = \frac{1}{4(2\pi)^2}\sum_{a,b=1}^{3}\int\frac{d^3p_a}{2p_a^0}\,\frac{d^3p_b}{2p_b^0} \\ \Big\{T^{\star ab,ik}(s,\vartheta_s)\,M^{abl}(s,t',u')\,\delta(p_a+p_b-p_1-p_2) \\ +T^{\star ab,kl}(t,\vartheta_t)\,M^{iab}(s',t,u')\,\delta(p_a+p_b-p_2-p_3) \\ +T^{\star ab,il}(u,\vartheta_u)\,M^{akb}(s',t',u)\,\delta(p_a+p_b-p_3-p_1)\Big\}. \end{aligned}$$

The three contributions in braces stem from the different possibilities of having one pion being a spectator not taking part in the final state interaction. The subscript d indicates that only the disconnected part has been considered and the angles ϑ_s, ϑ_t and ϑ_u denote the scattering angles in the corresponding channel.
We leave now the case where i, k and l are arbitrary and turn to $\eta \to \pi^+\pi^-\pi^0$.

The fact that $M^{+-0} = M^{1,1,3}$ yields some simplification: we may choose $i = k = 1$, $l = 3$. The above equation then becomes

$$\operatorname{Im} M(s,t,u)\Big|_d = \frac{1}{64\pi^2}\sum_{a,b=1}^{3}\int\frac{d^3p_a}{p_a^0\cdot p_a^0}\Big\{T^{\star ab,11}(s,\vartheta_s)M^{ab3}(s,t',u') + T^{\star ab,13}(t,\vartheta_t)M^{1ab}(s',t,u') + T^{\star ab,13}(u,\vartheta_u)M^{a1b}(s',t',u)\Big\}. \quad (2.92)$$

It is favorable to evaluate each of the three contributions in the CMS of the corresponding two particles undergoing a scattering. Let's stick to the case where π^0 is a spectator and evaluate the first term. Without loss of generality, we may then choose the angle ϑ appearing in $\int d\Omega \ldots = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \ldots$ to denote the angle between $\vec{p}_3$ and $\vec{p}_a$.
For further use let's find the CMS-expression for the Mandelstam variables t and u in terms of s and $\vartheta_0 = \angle(\vec{p}_1, \vec{p}_3)$. In the CMS of particle one and two we find: $s = 4{p^0}^2$ and $t = 2m_\pi^2 + p^0 p_3^0 + 2\cos\vartheta_0\,|\vec{p}\,|\,|\vec{p}_3|$, respectively, where $p^0 = p_1^0 = p_2^0$. Energy and momentum conservation implies $(p_1+p_2+p_3)^2 = m_\eta^2$ and we get $p_3^0 = \frac{1}{2\sqrt{s}}\left(m_\eta^2 - m_\pi^2 - s\right)$. This finally leads to the relations

$$\begin{aligned} t(s,\cos\vartheta_0) &= \frac{1}{2}\big(3s_0 - s + \cos\vartheta_0\,\kappa(s)\big), \\ u(s,\cos\vartheta_0) &= \frac{1}{2}\big(3s_0 - s - \cos\vartheta_0\,\kappa(s)\big), \end{aligned} \quad (2.93)$$

where

$$\kappa(s) = \sqrt{\frac{s-4m_\pi^2}{s}}\cdot\sqrt{\Big\{(m_\eta+m_\pi)^2 - s\Big\}\Big\{(m_\eta-m_\pi)^2 - s\Big\}}\,. \quad (2.94)$$

Analogous expressions hold for the other channels. $M(s,t',u')$ thus depends only on s and on the integration variable $\cos\vartheta$. Also the angles ϑ_s, ϑ_t and ϑ_u may be expressed through s, t, u and θ. Therefore the integrals can be evaluated in the same way as in section (2.8.2). What remains is an integral over the space angle Ω. Inserting equation (2.91) for M^{ikl} and using expression (2.60) found for $T^{lm,ik}$, one can now start evaluating all the Kronecker-δ products and sums and gets the following three contributions to $\operatorname{Im} M(s,t,u)$:

- $\pi^+\pi^-$ final state interaction, s fixed:

$$\frac{1}{128\pi^2}\sqrt{\frac{s-4m_\pi^2}{s}}\times\int d\Omega\sum_{a,b=1}^{3}T^{\star ab,11}(s,\vartheta_s)M^{ab3}(s,t',u') = \frac{1}{4\pi}\int d\Omega\,\frac{1}{3}\Big\{\big[3M(s,t',u') + M(t',u',s) + M(u',s,t')\big]e^{-\delta_0(s)}\sin\delta_0(s) - \big[M(u',s,t') + M(t',u',s)\big]e^{-\delta_2(s)}\sin\delta_2(s)\Big\}. \quad (2.95)$$

- $\pi^-\pi^0$ final state interaction, t fixed:

$$\frac{1}{4\pi}\int d\Omega\sum_{a,b=1}^{3}T^{\star ab,13}(t,\vartheta_t)M^{1ab}(s',t,u') = \frac{1}{128\pi^2}\sqrt{\frac{t-4m_\pi^2}{t}}\times\int d\Omega\,\frac{1}{2}\Big\{3\cos\vartheta_t\big[M(u',s',t) - M(s',t,u')\big]e^{-\delta_1(t)}\sin\delta_1(t) + \big[M(u',s',t) + M(s',t,u')\big]e^{-\delta_2(t)}\sin\delta_2(t)\Big\}. \quad (2.96)$$

- $\pi^+\pi^0$ final state interaction, u fixed:

$$\frac{1}{4\pi}\int d\Omega \sum_{a,b=1}^{3} T^{\star ab,13}(u,\vartheta_u)M^{a1b}(s',t',u) =$$
$$\frac{1}{128\pi^2}\sqrt{\frac{u-4m_\pi^2}{u}}\times\int d\Omega\,\frac{1}{2}\Big\{3\cos\vartheta_u\big[M(t',u,s')-M(s',t',u)\big]e^{-\delta_1(u)}\sin\delta_1(u)$$
$$+\big[M(t',u,s')+M(s',t',u)\big]e^{-\delta_2(u)}\sin\delta_2(u)\Big\}. \quad (2.97)$$

We have evaluated the three contributions in different frames. Since they are Lorentz invariant this procedure causes no problem and summing them up will yield the right hand side of equation (2.92). Unitarity has led us to a representation of $\mathrm{Im}\,M(s,t,u)$ which exclusively involves the function $M(s,t,u)$ itself and the phase shifts $\delta_I(s)$ describing elastic $\pi\pi$-scattering. Inserting (2.90) for $M(s,t,u)$ we can write the right hand side of (2.92) in the form

$$F_0(s)+(s-u)F_1(t)+(s-t)F_1(u)+F_2(t)+F_2(u)-\frac{2}{3}F_2(s).$$

Identifying corresponding terms on each side, we find expressions relating the functions $M_0(s)$, $M_1(s)$ and $M_2(s)$ among themselves. For $I=0$ e.g. we get

$$\mathrm{Im}\,M_0(s)\Big|_d = F_0(s) = \frac{1}{4\pi}\int d\Omega\, e^{-i\delta_0(s)}\sin\delta_0(s)$$
$$\left\{M_0(s)+\frac{2}{3}M_0(t')+\frac{2}{3}(s-u')M_1(t')+\frac{2}{3}(s-t')M_1(u')+\frac{20}{9}M_2(t')\right\}. \quad (2.98)$$

In the following we will write z instead of $\cos\vartheta$ for simplicity. Replacing $(s-u')$ by $(2s-3s_0+t')$ and $(s-t')$ by $(2s-3s_0+u')$ in (2.98) and making use of

$$\int\frac{d\Omega}{2\pi}f(t') = \int_{-1}^{1}dz\, f\left(\frac{3s_0-s+z\,\kappa(s)}{2}\right) \text{ and } \int d\Omega\, z^n f(t') = (-1)^n\int d\Omega\, z^n f(u'),$$

$\mathrm{Im}\,M_0(s)\Big|_d$ may be expressed in terms of angular averages of the form

$$\langle z^n f\,\rangle(s) = \frac{1}{2}\int_{-1}^{1}dz\, z^n f\left(\frac{3s_0-s+z\,\kappa(s)}{2}\right). \quad (2.99)$$

This reasoning holds also for $I=1$ and $I=2$. The imaginary parts of $M_I(s)$ can be represented in the following way:

$$\mathrm{Im}\,M_I(s)\Big|_d = \Big\{M_I(s)+\hat{M}_I(s)\Big\}\,e^{-i\delta_I(s)}\sin\delta_I(s). \quad (2.100)$$

The inhomogenities $\hat{M}_I(s)$ are

$$\begin{aligned}
\hat{M}_0 =& \frac{2}{3}\langle M_0\rangle + 2(s-s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}\kappa\langle zM_2\rangle,\\
\hat{M}_1 =& \frac{1}{\kappa}\left\{3\langle zM_0\rangle + \frac{9}{2}(s-s_0)\langle zM_1\rangle - 5\langle zM_2\rangle + \frac{3}{2}\kappa\langle z^2M_2\rangle\right\}, \quad (2.101)\\
\hat{M}_2 =& \langle M_0\rangle - \frac{3}{2}(s-s_0)\langle M_1\rangle(s) + \frac{1}{3}\langle M_2\rangle - \frac{1}{2}\kappa\langle zM_1\rangle,
\end{aligned}$$

where the arguments of $\langle M_0\rangle(s)$, ... have been omitted for simplicity.

2.11 Unitarity constraint and the one-loop χPT result

In order to get some confidence in the relations between the functions $M_I(s)$ and their imaginary parts found in the last section, it is a good idea to check whether they are indeed satisfied by the χPT one-loop result.
Let us first calculate the treelevel expressions for the phase shifts. The current algebra result for the transition amplitude $\pi^i(p_1)\pi^k(p_2) \to \pi^l(p_3)\pi^m(p_4)$ is

$$T_{\pi\pi}(s_{\pi\pi}, t_{\pi\pi}, u_{\pi\pi}) = \delta^{ik}\delta^{lm}T_1(s_{\pi\pi}) + \delta^{il}\delta^{km}T_2(t_{\pi\pi}) + \delta^{im}\delta^{kl}T_3(u_{\pi\pi}),$$

where $T_1(s_{\pi\pi}) = \frac{s_{\pi\pi}-m_\pi^2}{F_\pi^2}$, $T_2(t_{\pi\pi}) = T_1(t_{\pi\pi})$, $T_3(u_{\pi\pi}) = T_1(u_{\pi\pi})$. We have used the subscript $\pi\pi$ in order to distinguish the Mandelstam variables of $\pi\pi$-scattering from those of the decay $\eta \to 3\pi$.
From equations (2.53) and (2.61) we find, approximating $\sin\delta \cdot \exp(i\delta)$ by δ,

$$\begin{aligned}
\delta_0^{\rm tree}(s) &= \frac{\sqrt{1-4m_\pi^2/s}}{32\pi F_\pi^2}\left(2s - m_\pi^2\right) + \mathcal{O}\left(p^4\right) \\
\delta_1^{\rm tree}(s) &= \frac{\sqrt{1-4m_\pi^2/s}}{96\pi F_\pi^2}\left(s - 4m_\pi^2\right) + \mathcal{O}\left(p^4\right) \\
\delta_2^{\rm tree}(s) &= -\frac{\sqrt{1-4m_\pi^2/s}}{32\pi F_\pi^2}\left(s - 2m_\pi^2\right) + \mathcal{O}\left(p^4\right).
\end{aligned} \tag{2.102}$$

We have used $s_{\pi\pi} + t_{\pi\pi} + u_{\pi\pi} = 4m_\pi^2$ and $t_{\pi\pi} - u_{\pi\pi} = (s_{\pi\pi} - 4m_\pi^2)\cos\vartheta/2$.
At treelevel the amplitudes describing the decay $\eta \to 3\pi$ are: $M_0^{\rm tree}(s) = T(s) = (3s - 4m_\pi^2)/(m_\eta^2 - m_\pi^2)$, $M_1^{\rm tree}(s) = M_2^{\rm tree}(s) = 0$. To get $\hat{M}_I^{\rm tree}(s)$ we thus only need to know the angular averages $\langle T\rangle(s)$ and $\langle zT\rangle(s)$:

$$\langle T\rangle(s) = \frac{3 - T(s)}{2}, \qquad \langle zT\rangle(s) = \frac{\kappa(s)}{2(m_\eta^2 - m_\pi^2)}. \tag{2.103}$$

This yields, in view of equation(2.101):

$$\begin{aligned}
\hat{M}_0^{\rm tree} &= \frac{3 - T(s)}{3}, \\
\hat{M}_1^{\rm tree} &= \frac{3}{2\left(m_\eta^2 - m_\pi^2\right)} \\
\hat{M}_2^{\rm tree} &= \frac{3 - T(s)}{2}.
\end{aligned} \tag{2.104}$$

In terms of the treelevel expressions the unitarity constraint (2.100) reads

$$\mathrm{Im}\, M_I(s) = \left\{M_I^{\rm tree}(s) + \hat{M}_I^{\rm tree}(s)\right\}\delta_I^{\rm tree}(s) + \mathcal{O}\left(p^6\right). \tag{2.105}$$

So this equation should be satisfied when inserting the one-loop result ($\overline{M}_0$, $\overline{M}_1$, $\overline{M}_2$) on its left hand side. Using the explicit treelevel expressions (2.102) and (2.104) we get

$$\begin{aligned}
\mathrm{Im}\, M_0(s) &= \delta_0^{\rm tree}(s)\,\frac{3 + 2\,T(s)}{3} + \mathcal{O}\left(p^6\right), \\
\mathrm{Im}\, M_1(s) &= \delta_1^{\rm tree}(s)\,\frac{3}{2\left(m_\eta^2 - m_\pi^2\right)} + \mathcal{O}\left(p^6\right), \\
\mathrm{Im}\, M_2(s) &= \delta_1^{\rm tree}(s)\,\frac{3 - T(s)}{2} + \mathcal{O}\left(p^6\right).
\end{aligned} \tag{2.106}$$

Indeed, the quantities $\mathrm{Im}\,\overline{M}_I(s)$, given by the one-loop result for $\eta \to 3\pi$ presented in section 2.9, yield the right hand sides of equation (2.106).
This relation holds also to next order:

$$\mathrm{Im}\, M_I^{\text{2-loop}}(s) = \left\{ M_I^{\text{1-loop}}(s) + \hat{M}_I^{\text{1-loop}}(s) \right\} \times e^{-i\delta_I^{\text{1-loop}}(s)} \sin\delta_I^{\text{1-loop}}(s) + \mathcal{O}\left(p^8\right). \tag{2.107}$$

2.12 Discontinuities, imaginary parts and integration paths

As noted above, we have taken into account only the disconnected part of the T-matrix to determine the discontinuity of $M_I(s)$. Clearly there arises also a contribution due to the connected part:

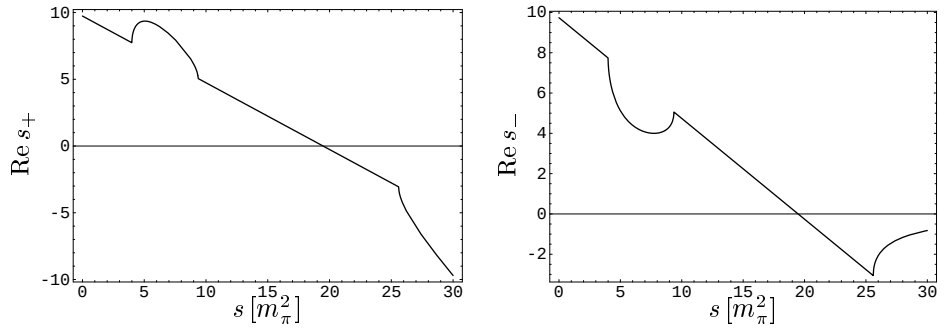
$$\mathrm{Im}\, M_I(s) = \mathrm{Im}\, M_I(s)\,|_d + \mathrm{Im}\, M_I(s)\,|_c\ .$$

The result for the discontinuity derived in section 2.10 agrees with the one-loop formula of chiral perturbation theory — considering a connected contribution to the T-matrix, arising by successive two-body collisions among different pairs of pions, would automatically give rise to two or more loops. At higher orders of the expansion the decay amplitude becomes complex and the angular averages $\hat{M}_I(s)$ will do so, too. Here we are running into a problem:

$$\mathrm{Im}\left(e^{i\delta}\mathrm{Im}M\right) = \sin\delta\,\mathrm{Im}M = \mathrm{Im}\left(\sin\delta(M+\hat{M})\right) = \sin\delta(\mathrm{Im}M + \mathrm{Im}\hat{M})$$
$$\Rightarrow\ \mathrm{Im}\hat{M} \equiv 0.$$

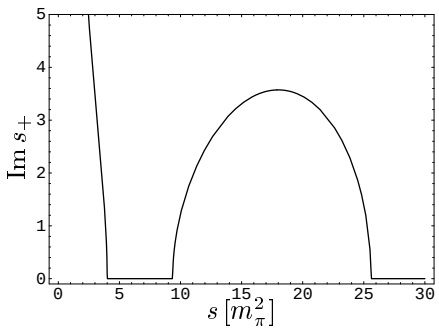
It is therefore not only an approximation, it is inconsistent with unitarity to simply drop the connected part of the T-matrix. What can be done? Consider again elastic $\pi\pi$-scattering. Once more, we ignore partial waves with $\ell \geq 2$. The scattering amplitude for this problem has the same structure as $A_{\eta\to 3\pi}$. This is plausible when replacing the η by a pion and transferring the isospin quantum numbers of the operator $\frac{1}{2}(m_u - m_d)\bar{q}\lambda^3 q$ to the fourth particle, which must be a π^0 then. There is one important simplification compared to $\eta \to 3\pi$: in the definition of $\kappa(s)$ we have to replace m_η by m_π and so $\kappa(s) \to (4m_\pi^2 - s)$. Therefore the quantity $\frac{3}{2}s_0 - \frac{1}{2}s + \frac{1}{2}z\kappa(s)$ becomes equal to $\frac{1}{2}(4m_\pi^2 - s)(1-z)$. When calculating the angular averages, one only meets values of M_I lying on the negative axis — i.e. we get only real contributions. So, in the case of elastic $\pi\pi$-scattering, it is consistent with unitarity to drop the connected contribution and to put $\mathrm{Im}M(s,t,u) = \mathrm{Im}M(s,t,u)\,|_d$ (extended elastic unitarity). The fact that we do not meet an inconsistency in the case of elastic $\pi\pi$-scattering points a way to resolve the problem occurring in $\eta - 3\pi$ decay: We make use of analytic continuation in the masses of the particles, namely the η. This means, we replace m_η by $\tilde{m}_\eta$ and start by putting the mass $\tilde{m}_\eta = m_\pi$, where we clearly know what we're doing. Consider the amplitude as a function of $\tilde{m}_\eta$ and extend the latter to other, also complex values, different from m_π.
The functions $M_I(s)$ are equipped with a cut starting at $4m_\pi^2$ and running along the real axis to ∞. The physical values are to be found on the upper rim of the cut. This cut is responsible for a second problem. Turning back to our equations for the $M_I(s)$'s in the case $\tilde{m}_\eta = m_\eta$ we see that, z running from -1 to +1, the argument $\frac{3}{2}s_0 - \frac{1}{2}s - \frac{1}{2}z\kappa(s)$ intersects the cut for certain values of s. So we have to deform the integration path in such a way that it stays away from the cut. In the following figures the real and imaginary parts of $s_+ = \frac{3}{2}s_0 - \frac{1}{2}s + \frac{1}{2}\kappa(s)$ and $s_- = \frac{3}{2}s_0 - \frac{1}{2}s - \frac{1}{2}\kappa(s)$ are shown in units of m_π^2. This makes it easy to read off

which integration paths are tending to run across the forbidden cut. The original paths are straight lines with endpoints s_- and s_+.

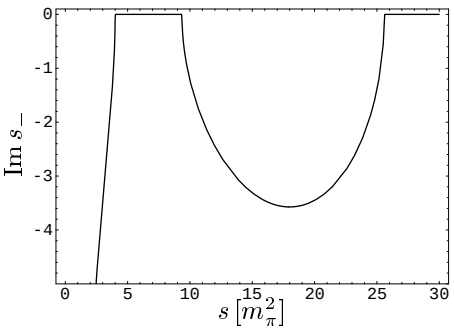


(a) Real part of s_+

(b) Real part of s_-



(c) Imaginary part of s_+



(d) Imaginary part of s_-

Figure 2.3: Real and imaginary parts of the endpoints of the integration path.

We make the following observations for the physical values of s ($s \geq 4m_\pi^2$):

- If $s > (m_\eta + m_\pi)^2$ both, s_+ and s_-, are real and less than $4m_\pi^2$ — no problem in choosing an appropriate integration path is encountered: we may just follow the straight line connecting the two endpoints.

- For values of s between $(m_\eta - m_\pi)^2$ and $(m_\eta + m_\pi)^2$ the endpoints s_+ and s_- become complex. As long as s is above $m_\eta^2 - 5m_\pi^2$ the real parts are less than $4m_\pi^2$ and the path avoids the cut. But as soon as s falls below this value, the straight line between s_+ and s_- hits the cut (s_+ lies in the upper and s_- in the lower half-plane) — the inconsistency arising from neglecting the connected part shows up here. We are forced now to deform the integration path. E.g. one may start at s_-, follow a vertical line approaching the lower rim, then turn to the left, advance towards the start point of the cut, encircle it and go back to the right, this time on the upper rim. For the last part of the track again move along a vertical line leading from the upper rim to s_+.

- Approaching $s = (m_\eta - m_\pi)^2$, the endpoints move close to each other to meet on the cut. It is important to note here, that they still lie on opposite rims of the cut, so that in the interval $\frac{1}{2}(m_\eta^2 - m_\pi^2) < s < (m_\eta - m_\pi)^2$, where $\kappa(s)$ is real, we still have to do the whole trip described above. At $s = \frac{1}{2}(m_\eta^2 - m_\pi^2)$

s_- becomes equal to $4m_\pi^2$ and moves on the upper half-plane. From now on, the integration path is given by a horizontal line on the upper rim of the cut.

For real endpoints bigger than $4m_\pi^2$, the problem to determine on which side of the cut they are situated occurs. The solution is given by equipping the η mass with a small positive imaginary part: $m_\eta \to m_\eta + i\delta$. Doing so, s_+ and s_- show the behaviour discussed above. When continuing the values of s into the unphysical region ($s < 4m_\pi^2$), the shape of the cut must be deformed – but this won't bother us. The complex m_η has the general effect of keeping the endpoints away from the cut. Note that the angular averages are taken over the values of $M_I(s)$ on the upper rim of the cut.

2.13 Integral equations

In this section we want to discuss how to solve the relations between M_0, M_1 and M_2 we have found from the unitarity condition. The discontinuities of the functions M_I are

$$\operatorname{disc} M_I(s) = \theta(s-4m_\pi^2)\Big\{M_I(s) + \hat{M}_I(s)\Big\}\sin\delta_I(s)e^{-i\delta_I(s)}. \tag{2.108}$$

On the right hand side it is understood to take the values on the upper rim of the cut: $M(s) \doteq M(s+i\epsilon)$, $\hat{M}(s) \doteq \hat{M}(s+i\epsilon)$.
To start consider a function $m(s)$ being analytic except for a cut along the real axis starting at $4m_\pi^2$. Assume $m(s)$ to remain finite as $s \to \infty$ and that the discontinuity across the the cut is given by

$$\operatorname{disc} m(s) = \theta(s-4m_\pi^2)\, m(s)\, \sin\delta(s)e^{-i\delta(s)}, \tag{2.109}$$

where the phase $\delta(s)$ is supposed to be known.
The homogeneous equation, written in terms of $m(s)$, reads

$$\frac{m(s+i\epsilon) - m(s-i\epsilon)}{2i} = \theta(s-4m_\pi^2)\, m(s+i\epsilon)\, \sin\delta(s)e^{-i\delta(s)}$$

which yields

$$m(s+i\epsilon) = e^{2i\delta(s)}m(s-i\epsilon).$$

Taking the logarithm on each side, this leads to

$$\operatorname{disc}\ln(m(s)) = \delta(s)\,\theta(s-4m_\pi^2).$$

We may now apply the Cauchy integral representation discussed in appendix A.4 to this equation. Assuming that $\delta(s)$ remains finite as $s \to \infty$, we get

$$\ln m(z) = p + \frac{s}{2i\pi}\int_{4m_\pi^2}^{\infty} \frac{\ln m(s'+i\epsilon) - \ln m(s'-i\epsilon)}{s'(s'-z)}ds' = p + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta(s')}{s'(s'-z)}ds'.$$

Setting $p = 0$ we find the function

$$\Omega(s) \doteq \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta(s')}{s'(s'-s)}ds'\right\}, \tag{2.110}$$

referred to as the Omnès function, which is normalized to $\Omega(0) = 1$. $|\Omega(s)|$ is continuous and the phase above and below the cut is given by $\exp(i\delta)$ and $\exp(-i\delta)$,

respectively. Besides that, $\Omega(s)$ has got no zeros, so that $Q(s) = m(s)/\Omega(s)$ is an entire function.
We might argue from the above expression for $\ln m(z)$ that $m(s)$ has to be of the form $m(s) = c \cdot \Omega(s)$. In general this needs not to be true, as we are going to show now.
Assume that the phase shift δ is equal to zero at $s = 4m_\pi^2$, rises to $\alpha\pi$ at $s = \Lambda^2$ and stays there for larger s. In the case where α is integer, the cut is only present for $4m_\pi^2 \leq s \leq \Lambda^2$. For $s \geq \Lambda^2$ we may evaluate the integral over the phase shift: $\int_{4m_\pi^2}^{\infty} = \int_{4m_\pi^2}^{\Lambda^2} + \int_{\Lambda^2}^{\infty}$. The first integral tends to $-\frac{1}{\pi}\int_{4m_\pi^2}^{\Lambda^2} \frac{ds'\delta(s')}{s'}$ as $s \to \infty$, whereas the the second one becomes

$$\frac{s}{\pi}\int_{\Lambda^2}^{\infty} \frac{ds'\delta(s')}{s'(s'-s)} = \alpha \int_{\Lambda^2}^{\infty} ds' \left(\frac{1}{s'-s} - \frac{1}{s'}\right) = -\alpha \ln\left(\frac{\Lambda^2 - s}{\Lambda^2}\right).$$

The asymptotic behaviour of $\Omega(s)$ is therefore determined by α:

$$\Omega(s) \asymp \left(\frac{\Lambda^2 - s}{\Lambda^2}\right)^{-\alpha}. \tag{2.111}$$

The notation $f(x) \asymp g(x)$ means that $f(x)$ is asymptotically equal to $g(x)$.
We conclude that $Q(s)$ doesn't grow faster than s^α, because we required $m(s)$ to stay finite in the limit $s \to \infty$.
Since the condition $\operatorname{disc} m(s) = \theta(s-4m_\pi^2)\, m(s+i\epsilon) \sin\delta(s) e^{-i\delta(s)}$ is linear in m(s), multiplying $\Omega(s)$ by any entire function $f(z)$ will yield a solution to the homogeneous equation. $m(s)$ must therefore be of the form

$$m(s) = \Omega(s) \sum_{i=0}^{n} c_i s^i \quad \text{with } n \leq \alpha.$$

We now turn back to the inhomogeneous equation and assume $\hat{M}(s)$ to be given. Consider again the function $Q(s) = M(s)/\Omega(s)$ and its discontinuity (we leave the factor $\theta(s-4m_\pi^2)$ implicit):

$$\begin{aligned}
\operatorname{disc} Q(s) &= \frac{M(s+i\epsilon)}{2i\Omega(s+i\epsilon)} - \frac{M(s-i\epsilon)}{2i\Omega(s-i\epsilon)} \\
&= \frac{M(s+i\epsilon)e^{-i\delta(s)} - M(s-i\epsilon)e^{+i\delta(s)}}{2i|\Omega(s)|} \\
&= \frac{e^{i\delta(s)}\operatorname{disc} M(s)}{|\Omega(s)|} - \frac{\sin\delta(s) M(s)}{|\Omega(s)|} \\
&= \frac{\sin\delta(s)\hat{M}(s)}{|\Omega(s)|} \cdot \theta(s-4m_\pi^2).
\end{aligned}$$

Invoking the the Cauchy integral representaion for $Q(s)$ we get:

$$Q(s) = \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\sin\delta(s')\hat{M}(s')}{|\Omega(s')|s'^n(s'-s)} ds' + P(x), \tag{2.112}$$

where $P(x)$ is a subtraction polynomial of order $n-1$. Requiring $M(s,t,u)$ to grow at most linearly in all directions as s, t, $u \to \infty$ implies:

$$M_0(s) = \mathcal{O}(s), \quad M_1 = \mathcal{O}(1), \quad M_2 = \mathcal{O}(s).$$

The number of required subtractions is then determined by the asymptotic behaviour of the Omnès factors. If one uses a parametrisation of the phase shifts in such a way that $\delta_0(s)$ and $\delta_1(s)$ tend to π for $s \to \infty$ and $\delta_2(s)$ asymptotically tends to zero, the subtraction polynomials must be:

$$\begin{aligned} P_0(s) &= \alpha_0 + \beta_0 s + \gamma_0 s^2, \\ P_1(s) &= \alpha_1 + \beta_1 s, \\ P_2(s) &= \alpha_2 + \beta_2 s. \end{aligned}$$

A term $\gamma_2 s^2$ in $P_2(s)$ would give the wrong asymptotic behaviour of $M_2(s)$ as $|s| \to \infty$ – therefore $\gamma_2 \equiv 0$. Three of the seven subtraction constants may be eliminated by the observation that the decomposition of the function $M(s,t,u)$ into the three components $M_0(s)$, $M_1(s)$ and $M_2(s)$ is not unique. The transformation

$$\begin{aligned} M_0(s) &\to M_0(s) + 3c_1\,(s - s_0) + \frac{4}{3}c_2 + c_3\left(3s_0 - \frac{5}{3}s\right), \\ M_1(s) &\to M_1(s) + c_1, \\ M_2(s) &\to M_2(s) + c_2 + c_3 s \end{aligned} \tag{2.113}$$

yields the same total amplitude. This allows us to set $\alpha_1 = \alpha_2 = \beta_2 = 0$. The four remaining subtraction constants may be determined with the aid of the one-loop result. The discussion thereof follows later.
We finally get the following form of the dispersion integrals:

$$\begin{aligned} M_0(s) &= \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_0(s')\hat{M}_0(s')}{|\Omega_0(s')|\,(s'-s-i\epsilon)}\right), \\ M_1(s) &= \Omega_1(s)\left(\beta_1 s + \frac{s}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1(s')\hat{M}_1(s')}{|\Omega_1(s')|\,(s'-s-i\epsilon)}\right), \\ M_2(s) &= \Omega_2(s)\,\frac{s^2}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_2(s')\hat{M}_2(s')}{|\Omega_2(s')|\,(s'-s-i\epsilon)}. \end{aligned} \tag{2.114}$$

This set of coupled integral equations is to be solved numerically. The numerics is described in chapter 3.
We have introduced the Omnès functions and formulated the dispersion relation for $Q(s) = M(s)/\Omega(s)$ (equation(2.112)). This seems to be a detour. Why not write down the relation

$$M(s) = \frac{s}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta(s')e^{-i\delta(s')}\{M(s') + \hat{M}(s')\}}{s'-s} + c\,, \tag{2.115}$$

which follows directly from the unitarity constraint, and solve this equation iteratively? (We have assumed $M = \mathcal{O}(1)$ for $|s| \to \infty$.) Let us remember the solution to the homogeneous equation. In the case where the phase shift δ asymptotically approaches π, we have found $m(s)$ to be of the form $m(s) = (c_0 + c_1 s)\Omega(s)$. This solution involves two constants to be determined. The corresponding dispersion relation, however, contains only one free constant, which fixes the value of $m(s)$ at the origin: $m(0) = c_0$. The parameter c_1 remains free. This problem occurs also in the case of the inhomogeneous equation (2.115) – an attempt to solve it iteratively is doomed to failure. Removing the first term in the unitarity condition

$\operatorname{disc} M = \sin\delta e^{-i\delta}\{M + \hat{M}\}$ by introducing Q, resolves the problem of ambiguity. Assuming the inhomogenities in (2.114) to be known, the iteration process will converge. The numerical results indicate that this is also true when the inhomogenities depend on the functions M_I and thus the three equations (2.114) get coupled.
Next we will consider how to determine the subtraction constants and the representation of the phase shifts.

2.14 Determination of the subtraction constants

The subtraction constants are determined by matching the dispersive result to the χPT one-loop amplitude.
We have a certain freedom determining the number of subtraction constants. $\pi\pi$-scattering indicates that two subtraction should be sufficient – the chiral perturbation series for $\eta \to 3\pi$ thus leads to an oversubtracted form. However, using only two subtractions would require to look closer at the contributions above $K\bar{K}$-threshold. On the other hand, introducing too many subtractions will just reproduce the one loop result, when using this one to fix the subtraction constants. As already mentioned in section 2.13, the number of subtractions may be fixed by the requirement that $M(s,t,u)$ grows at most linearly in all directions as s, t, $u \to \infty$ – we have to determine four subtraction constants.
In the limit $m_u = m_d = 0$ the amplitude exhibits two Adler zeros, one at $p_{\pi^+} = 0$ and the other at $p_{\pi^-} = 0$. This corresponds to $s = u = 0$ and $s = t = 0$, respectively. The Adler zeros are a consequence of $SU(2)_R \times SU(2)_L$ symmetry and do not rely on the expansion in m_s. In the case of a vanishing pion mass m_π, the treelevel prediction for the amplitude vanishes along the line $s = 0$. Higher order terms will generate an imaginary part and distort the line $\operatorname{Re} M(s,t,u) = 0$. However, if the quark masses m_u and m_d vanish, this line still passes through the two Adler zeros specified above.
If we turn on the quark masses, a curve along which the real part of the amplitude vanishes is still present, but shifted away from the points $s = u = 0$ and $s = t = 0$. This shift, however, is an $\mathcal{O}(m_u, m_d)$ effect. Higher order contributions will change the shape of the curve only slightly in the vicinity of the points $s = u = 0$ and $s = t = 0$.
The line described by $u = s$, $t = 3s_0 - 2s$ intersects the curve of vanishing real part at some point. At leading order, this happens at $s = s_A = \frac{4}{3}m_\pi^2$. The one-loop corrections shift this point up to $s_A = 1.41\, m_\pi^2$. Due to the above arguments we may assume that higher order corrections won't affect the position of the intersection significantly any more. This is confirmed by the fact that already the leading correction to the treelevel prediction of s_A is of about only 6%. The current algebra and the one-loop results are very close to each other in the vicinity of the Adler zero – the slope differs by less than a percent. (Note, that the slope, however, is not protected by $SU(2) \times SU(2)$ symmetry.)
For these reasons we assume the one-loop prediction to be reliable in the vicinity of $s = s_A$, and require the first two terms of the Taylor series of $M(s,t,u)$ around s_A to agree with those of $\overline{M}(s,t,u)$ along the line $s = u$:

$$M(s, 3s_0 - 2s, s) = M_A + (s - s_A)S_A + \mathcal{O}\left((s - s_A)^2\right) . \tag{2.116}$$

M_A and S_A are given by

$$M_A = -0.0041\,\mathrm{i} \quad , \qquad S_A = (0.1957 - 0.0519\,\mathrm{i})\, m_\pi^{-2} . \tag{2.117}$$

This fixes the two subtraction constants α_0 and β_0. After each iteration step corrections to α_0 and β_0 are calculated such, that equation (2.116) is satisfied.

To fix β_1 and γ_0 we first look at the dispersion relations satisfied by the one-loop result. For $s \to \infty$, $\overline{M}_0(s)$, $\overline{M}_1(s)$ and $\overline{M}_2(s)$ show the following asyptotic behaviour:

$$\begin{aligned}
\overline{M}_0(s) &\asymp -\frac{1}{2}\cdot\frac{s^2\ln s - \sigma s^2}{16\pi^2 F_\pi^2\left(m_\eta^2 - m_\pi^2\right)}\,,\\
\overline{M}_1(s) &\asymp -\frac{3}{8}\cdot\frac{s\ln s - \sigma s - s\left(4L_3 + 1/64\pi^2\right)}{16\pi^2 F_\pi^2\left(m_\eta^2 - m_\pi^2\right)}\,,\\
\overline{M}_2(s) &\asymp +\frac{3}{8}\cdot\frac{s^2\ln s - \sigma s^2}{16\pi^2 F_\pi^2\left(m_\eta^2 - m_\pi^2\right)}\,,
\end{aligned} \tag{2.118}$$

where $\sigma = \ln\mu^2 + 1 + i\pi$.
From this we conclude that $\overline{M}_0(s)$ and $\overline{M}_2(s)$ obey a dispersion relation with three subtractions, while two subtractions are sufficient for $\overline{M}_1(s)$:

$$\begin{aligned}
\overline{M}_0(s) &= a_0 + b_0 s + c_0 s^2 + \frac{s^3}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\mathrm{Im}\,\overline{M}_0(s')}{s'-s-i\epsilon}\,,\\
\overline{M}_1(s) &= a_1 + b_1 s + \frac{s^2}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\mathrm{Im}\,\overline{M}_1(s')}{s'-s-i\epsilon}\,,\\
\overline{M}_2(s) &= a_2 + b_2 s + c_2 s^2 + \frac{s^3}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\mathrm{Im}\,\overline{M}_2(s')}{s'-s-i\epsilon}\,.
\end{aligned} \tag{2.119}$$

From equation (2.118) we find for the two combinations $\overline{M}_0(s) + \frac{4}{3}\overline{M}_2(s)$ and $s\overline{M}_1(s) + \overline{M}_2(s)$:

$$\overline{M}_0(s) + \frac{4}{3}\overline{M}_2(s) \asymp \mathcal{O}\left(s\ln s\right) \quad \text{and} \quad s\overline{M}_1(s) + \overline{M}_2(s) \asymp s^2\cdot\frac{-4L_3 + 1/64\pi^2}{\left(m_\eta^2 - m_\pi^2\right)F_\pi^2}\,. \tag{2.120}$$

If the dispersive representations (2.119) shall reproduce the asymptotic behaviour of these two combinations, the terms $\mathcal{O}(s^2)$ arising from the subtraction polynomials must be compensated by those from the dispersion integrals. We conclude

$$c_0 + \frac{4}{3}c_2 = \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\left\{\mathrm{Im}\,\overline{M}_0(s') + \frac{4}{3}\mathrm{Im}\,\overline{M}_2(s')\right\} \tag{2.121}$$

and

$$b_1 + c_2 = -\frac{4L_3 - 1/64\pi^2}{\left(m_\eta^2 - m_\pi^2\right)F_\pi^2} + \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\left\{s'\mathrm{Im}\,\overline{M}_1(s') + \mathrm{Im}\,\overline{M}_2(s')\right\}. \tag{2.122}$$

When establishing the integral equations for the functions $M_I(s)$, we have found that they are determined only up to a polynomial (see (2.113)). This allowed us to eliminate three of seven subtraction constants. Using the representation (2.119), $\overline{M}(s,t,u)$ receives two pieces; one collecting the dispersion integrals, the other one being a combination of the subtraction polynomials, which is of the form $P(s,t,u) = a + bs + cs^2 - d\left(s^2 + 2tu\right)$, where $c = c_0 + \frac{4}{3}c_2$ and $d = c_0 + c_2$. To get this form of $P(s,t,u)$ one makes use of $s + t + u = 3s_0$. Thus also in the case of the one-loop result only four combinations of the eight subtraction constants are of physical significance.

The quantities $M_0''(0)$, $M_1'(0)$ and $M_2''(0)$ are independent of the convention (2.113) – but they are related to β_1 and γ_0:

$$\begin{aligned} M_1'(0) + \frac{1}{2}M_2''(0) &= \beta_1 + \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\left\{\frac{s'\sin\delta_1(s')\hat{M}_1 s'}{|\Omega_1(s')|} + \frac{\sin\delta_2(s')\hat{M}_2 s'}{|\Omega_2(s')|}\right\}, \\ M_0''(0) + \frac{4}{3}M_2''(0) &= 2\gamma_0 + \frac{2}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^3}\left\{\frac{\sin\delta_0(s')\hat{M}_0 s'}{|\Omega_0(s')|} + \frac{4}{3}\frac{\sin\delta_2(s')\hat{M}_2 s'}{|\Omega_2(s')|}\right\}. \end{aligned} \tag{2.123}$$

The corresponding expressions in one-loop approximation are of the same structure. They receive a contribution from an integral over the discontinuities of the amplitude and a subtraction term. Explicitly we have

$$\begin{aligned} &\overline{M}_1'(0) + \frac{1}{2}\overline{M}_2''(0) = b_1 + c_2\,, \\ \text{and}\qquad &\overline{M}_0''(0) + \frac{4}{3}\overline{M}_2''(0) = 2\left(c_0 + \frac{4}{3}c_2\right). \end{aligned} \tag{2.124}$$

The right hand sides of these equations are given by (2.121) and (2.122), respectively. Identifying the corresponding dispersion integrals with those in equation (2.123), we get for β_1 and γ_0:

$$\begin{aligned} \beta_1 &\approx -\frac{4L_3 - 1/64\pi^2}{\left(m_\eta^2 - m_\pi^2\right)F_\pi^2}\,, \\ \gamma_0 &\approx 0\,. \end{aligned} \tag{2.125}$$

With $L_3 = (-3.5 \pm 1.1)\cdot 10^{-3}$ the numeric value of β_1 becomes

$$\beta_1 = (6.5 \pm 1.8)\mathrm{GeV}^{-4}.$$

The main difference between the dispersive representation and the one loop approximation is, that the integrals in (2.123) include only $\pi\pi$-discontinuities, while the one in (2.121) and (2.122) also account for the singularities generated by $K\bar{K}$, $\pi\eta$ and $\eta\eta$ intermediate sates. This causes some uncertainties in the determination of β_1 and γ_0. As an estimate of these uncertainties one may consider the contribution to the dispersion integrals of the one-loop approximation above $K\bar{K}$-threshold – the one from $\pi\eta$ intermediate states is proportional to m_π^2 and thus small. The significant effects are therefore due to the interval $4m_K^2 < s < \infty$. Taking these contributions into account will shift the two subtraction constants up to

$$\beta_1 = 8.3\,\mathrm{GeV}^{-4} \quad \text{and} \quad \gamma_0 = 8.6\,\mathrm{GeV}^{-4}.$$

The difference between the integrals over the elastic region $4m_\pi^2 < s < (m_\eta + m_\pi)^2$ are of $\mathcal{O}\left(p^2\right)$, which is beyond the accuracy of the one loop prediction. However, within the elastic region the dispersive analysis will give the better representation, because it takes into account multiple two-particle rescattering. This justifies the presented way of determining the subtraction constants β_1 and γ_0. The one-loop contributions from inelastic channels are used only to estimate the uncertainties.

2.15 Input for the phase shifts $\delta_I(s)$

For the phase-shifts $\delta_0 = \delta_0^0$, $\delta_1 = \delta_1^1$ and $\delta_2 = \delta_2^0$ we use the following representation, due to A. Schenk:

$$\tan\delta_\ell^I = \sqrt{\frac{s-4m_\pi^2}{s}}\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)^\ell \frac{s_\ell^I - 4m_\pi^2}{s_\ell^I - s}\times \left\{a_\ell^I + \bar{b}_\ell^I\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right) + c_\ell^I\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)^2\right\}, \quad (2.126)$$

where

$$\bar{b}_\ell^I = b_\ell^I - a_\ell^I\frac{4m_\pi^2}{s_\ell^I - 4m_\pi^2} + \left(a_\ell^I\right)^3\delta_{0\ell}\,.$$

The Schenk-parameters are not known definitely. I have solved the dispersion relations for three sets of parameters. The first set is the same as used in [2] and is mainly given by the one-loop result.

- *Set A*

$a_0^0 = 0.217$	$a_0^2 = -0.042$	$a_1^1 = 0.037$
$b_0^0 = 0.240$	$b_0^2 = -0.075$	$b_1^1 = 0.005$
$c_0^0 = 0$	$c_0^2 = 0$	$c_1^1 = 0$
$s_0^0 = 0.748\,\mathrm{GeV}^2$	$s_0^2 = -0.469\,\mathrm{GeV}^2$	$s_1^1 = 0.591\,\mathrm{GeV}^2$

The other two sets are determined by threshold parameters obtained from the two-loop calculation, ref. [13], and correspond to set I and II in table 4 therein. According to J. Gasser one should trust rather set II than set I, because it gives the better result for the D-waves.

- *Set I*

$a_0^0 = 0.217$	$a_0^2 = -0.0413$	$a_1^1 = 0.0420$
$b_0^0 = 0.272$	$b_0^2 = -0.0701$	$b_1^1 = 0.00390$
$c_0^0 = -0.0210$	$c_0^2 = 0.000113$	$c_1^1 = 0.000297$
$s_0^0 = 0.755\,\mathrm{GeV}^2$	$s_0^2 = -0.877\,\mathrm{GeV}^2$	$s_1^1 = 0.599\,\mathrm{GeV}^2$

- *Set II*

$a_0^0 = 0.206$	$a_0^2 = -0.0443$	$a_1^1 = 0.0420$
$b_0^0 = 0.272$	$b_0^2 = -0.0703$	$b_1^1 = 0.00392$
$c_0^0 = -0.0210$	$c_0^2 = 0.000145$	$c_1^1 = 0.000324$
$s_0^0 = 0.754\,\mathrm{GeV}^2$	$s_0^2 = -0.877\,\mathrm{GeV}^2$	$s_1^1 = 0.598\,\mathrm{GeV}^2$

Here are the corresponding plots for the phase-shifts:

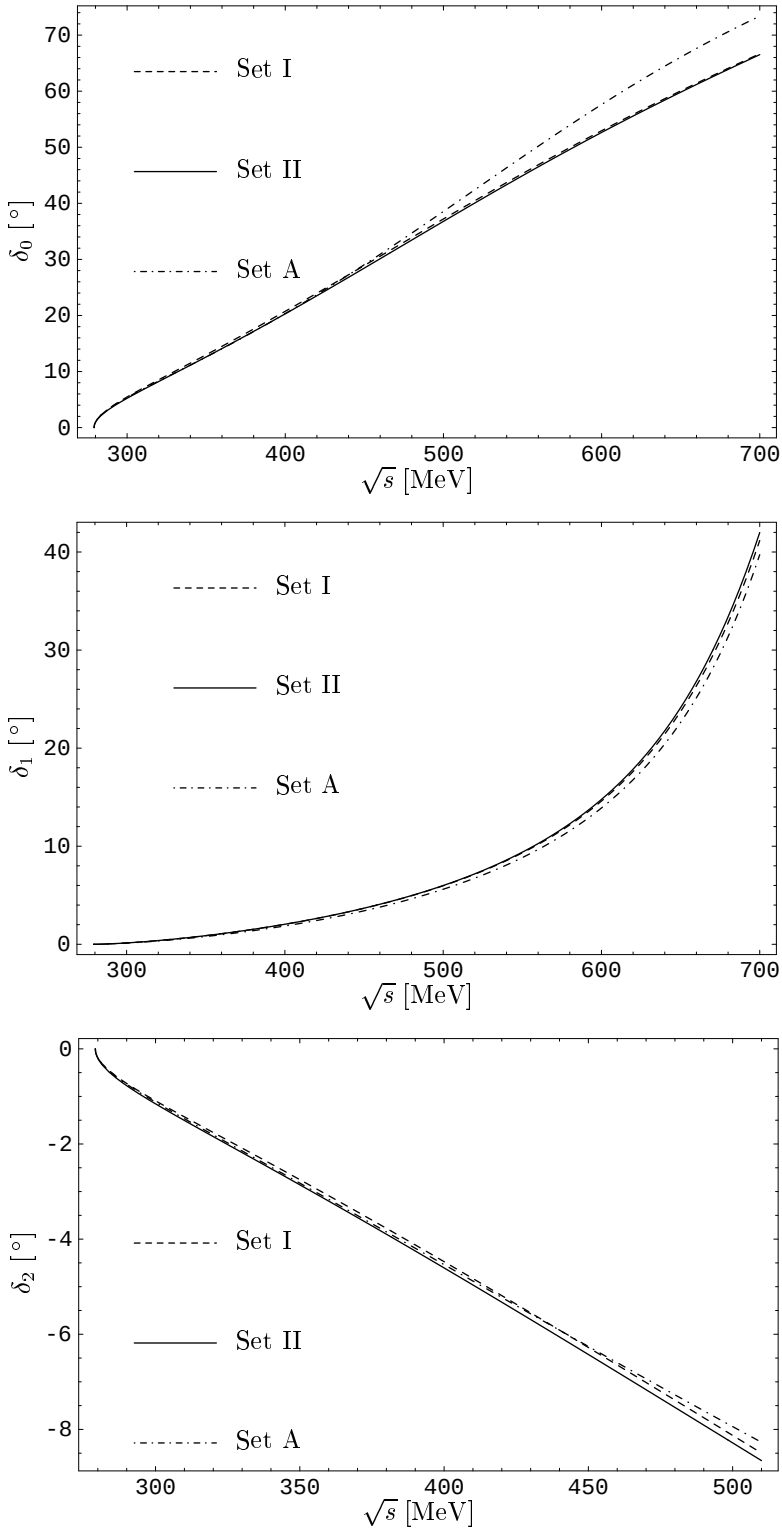


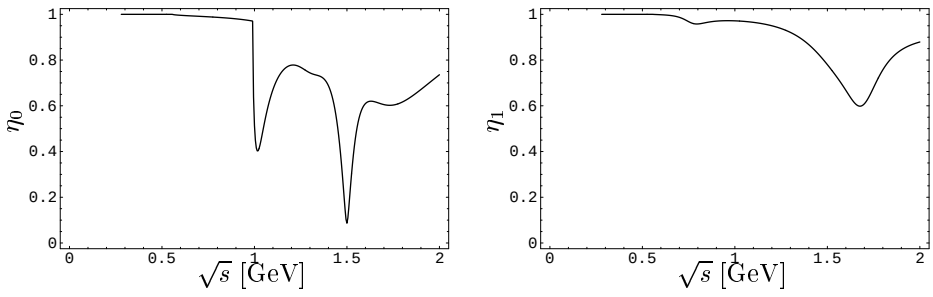
Figure 2.4: The phase shifts $\delta_0(s)$, $\delta_1(s)$ and $\delta_2(s)$.

2.16 Inelasticity in the $\pi\pi$ final state interaction

The dispersion relations (2.114) as they stand disregard the inelastic contributions to the imaginary parts. In this section we want to make an attempt to take the inelasticities associated with two particle rescattering into account. In the elastic region, the contributions to the total $\pi\pi$-scattering amplitude corresponding to each partial wave are proportional to $t_I = \sin\delta_I \exp i\delta_I = \frac{1}{2i}(\exp 2i\delta_I - 1)$. Inelasticity is taken care of by introdicing the inelasticity parameters η_I:

$$t_I = \frac{1}{2i}\left(\eta_I\, e^{2i\delta_I} - 1\right), \quad \text{where} \quad 0 \le \eta_I \le 1. \tag{2.127}$$

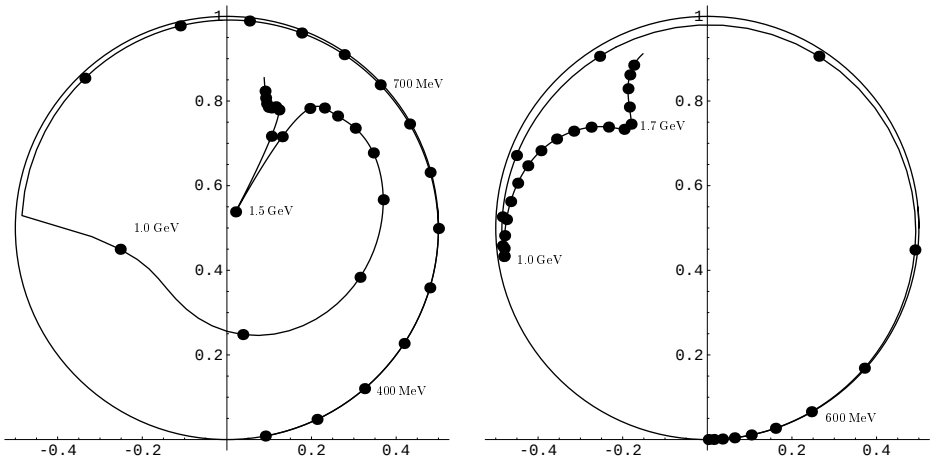
To get the numerical values of the parameters η_I, I used the output of a program, which was kindly put at disposal by J. Gasser. In that representation only η_0 and η_1 differ from 1:



(a) η_0 (b) η_1

Figure 2.5: The inelasticity parameters η_0 and η_1.

To get an idea how strong inelasticity causes the partial waves t_I to differ from the elastic case, it is convenient to draw an Argand plot:



(a) t_0 (b) t_1

Figure 2.6: Argand plot for $I = 0$ and $I = 1$. The phase shifts correspond to set II.

The $K\bar{K}$-threshold at 988 MeV shows up quite impressively for $I = 0$, but also the resonances $f_0(1500$ MeV), $\rho(770$ MeV) and $\rho(1700$ MeV) appear very clearly.
Neglecting the presence of inelasticity, i.e setting $\eta_I = 1$, forces $t_I(s)$ to lie on the unitarity circle. This will lead to an over estimation of final state effects and the decay rate – the solutions of the dispersion relations ignoring $\eta \neq 1$ obtained with set I or II are expected to give an upper bound for the rate and thus for the quark mass ratio Q. Nevertheless, the result corresponding to set I and II should be closer to the true value of Q than the prediction obtained with set A.

We now want to find modified integral equations according to the representation of the partial waves given in equation (2.127). We proceed in an analogous way as in the elastic case.
Equation (2.108) for the discontinuity of $M_I(s)$ has to be replaced by

$$\operatorname{disc} M_I(s) = \theta\left(s - 4m_\pi^2\right) \cdot \left\{ M_I(s) + \hat{M}_I(s) \right\} \frac{1 - \eta_I(s) e^{-2i\delta(s)}}{2i} \,. \tag{2.128}$$

The expressions (2.101) for the angular averages $\hat{M}_I(s)$ remain unchanged. Again we first solve the corresponding homogeneous equation

$$\operatorname{disc} m_I(s) = \theta\left(s - 4m_\pi^2\right) \cdot m_I(s) \frac{1 - \eta_I(s) e^{-2i\delta(s)}}{2i} \,.$$

The solution to this equation is given by the modified Omnès factor $\widetilde{\Omega}(s)$:

$$\widetilde{\Omega}(s) = \exp\left\{ \frac{s}{\pi} \int\limits_{4m_\pi^2}^{\infty} \frac{\delta(s') - \frac{1}{2i} \ln \eta(s')}{s'\,(s' - s)} \, ds' \right\} . \tag{2.129}$$

If $\eta(s) \to 1$ for $s \to \infty$, $\widetilde{\Omega}(s)$ has the same asymptotic behaviour as $\Omega(s)$.
For further use we introduce the Ξ-factor defined as

$$\Xi(s) = \exp\left\{ \frac{is}{2\pi} \; \mathcal{P} \int\limits_{16m_\pi^2}^{\infty} \frac{\ln \eta(s')}{s'(s' - s)} \, ds' \right\} . \tag{2.130}$$

For values of s that do not lie in the interval $[16m_\pi^2, \infty)$, the symbol $\mathcal{P}$ is understood to be ignored. The lower integration limit has been shifted to $16m_\pi^2$, since $\eta(s) \equiv 1$ for $s < 16m_\pi^2$. With the aid of the Ξ-factor we may now write

$$\widetilde{\Omega}(s) = \xi(s) \cdot \Xi(s) \cdot \Omega(s) \quad \text{where} \quad \xi(s) = \begin{cases} \eta^{-1/2}(s) & \text{on the upper rim,} \\ \eta^{1/2}(s) & \text{on the lower rim,} \\ 1 & \text{else.} \end{cases} \tag{2.131}$$

The analog to the function $Q(s)$ in section 2.13 is $\widetilde{Q}(s) = M(s)/\widetilde{\Omega}(s)$. Its discontinuity is

$$\operatorname{disc} \widetilde{Q}(s) = \theta\left(s - 4m_\pi^2\right) \cdot \frac{\left\{ \eta(s)^{-1/2} e^{i\delta(s)} - \eta(s)^{1/2} e^{-i\delta(s)} \right\} \cdot \hat{M}(s)}{2i\overline{\Omega}(s)} \,, \tag{2.132}$$

where $\overline{\Omega}(s) = \Xi(s) \cdot |\Omega(s)|$.
We now are in the position to write down the modified integral equations. They

are very similar to those displayed in (2.114). They only involve the additional functions $\eta_I(s)$, $\Xi_I(s)$ and $\xi(s)$:

$$\begin{aligned}
M_0(s) &= \xi_0(s)\cdot\Xi_0(s)\cdot\Omega_0(s)\times \\
&\left(\alpha_0+\beta_0 s+\gamma_0 s^2+\frac{s^2}{2i\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\,\frac{\left\{\eta_0^{-1/2}e^{i\delta_0}-\eta_0^{1/2}e^{-i\delta_0}\right\}\cdot\hat{M}_0(s')}{\Xi_0(s')\,|\Omega_0(s')|\,(s'-s-i\epsilon)}\right), \\
M_1(s) &= \xi_1(s)\cdot\Xi_1(s)\cdot\Omega_1(s)\times \\
&\left(\beta_1 s+\frac{s}{2i\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\,\frac{\left\{\eta_1^{-1/2}e^{i\delta_1}-\eta_1^{1/2}e^{-i\delta_1}\right\}\cdot\hat{M}_1(s')}{\Xi_1(s')\,|\Omega_1(s')|\,(s'-s-i\epsilon)}\right), \qquad (2.133) \\
M_2(s) &= \Omega_2(s)\,\frac{s^2}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\,\frac{\sin\delta_2\cdot\hat{M}_2(s')}{|\Omega_2(s')|\,(s'-s-i\epsilon)}\,.
\end{aligned}$$

The argument s' of the functions δ_I and η_I appearing in the integrals have been omitted for better readability. We have already mentioned that in our representation $\eta_2(s)\equiv 1$. The equation for $M_2(s)$ therefore remains unchanged. One readily checks that in the elastic limit, $\eta_I\equiv 1$, we get the original integral equations back. This serves as a consistency check.

3 Computational methods

In this chapter it will be discussed in detail how to solve the integral equations numerically.
Clearly it would have been possible to solve most of the problems using existing programs as e.g. *Mathematica*. However, I preferred to program everything by myself. This has the advantage to know exactly what the computer is doing and you are forced to really look at the algorithms. Last but not least one also learns, that writing such a program takes much more time than thought before.

3.1 Computation of the Omnès functions Ω_I

There are several ingredients belonging to the integral equations. Let's start with the Omnès functions, given by

$$\Omega_I(s) = \exp\left\{\frac{s}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s')}{s'(s'-s)}\,ds'\right\}. \tag{3.1}$$

First consider the case of real $s > 4m_\pi^2$. To get the values of the Omnès functions on the upper and lower rim of the cut, $\Omega_I(s \pm i\epsilon)$, we use the Sokhotsky-Plemelj formula:

$$\frac{s}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s')}{s'(s'-s\mp i\epsilon)}\,ds' = \pm i\delta_I(s) + \frac{s}{\pi}\,\mathcal{P}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta(s')}{s'(s'-s)}\,ds'.$$

The Cauchy principal value is treated as follows:

$$\frac{s}{\pi}\,\mathcal{P}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s')}{s'(s'-s)}\,ds' = \frac{s}{\pi}\,\mathcal{P}\left(\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s)}{s'(s'-s)}\,ds' + \int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s')-\delta_I(s)}{s'(s'-s)}\,ds'\right).$$

The first integral yields

$$\frac{s\cdot\delta_I(s)}{\pi}\lim_{\epsilon\to\infty}\left(\int\limits_{4m_\pi^2}^{s-\epsilon}\frac{ds'}{s'(s'-s)} + \int\limits_{s+\epsilon}^{\infty}\frac{ds'}{s'(s'-s)}\right) = \frac{\delta_I(s)}{\pi}\,\ln\left(\frac{4m_\pi^2}{s-4m_\pi^2}\right),$$

whereas the second one,

$$\frac{s}{\pi}\,\mathcal{P}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_I(s')-\delta_I(s)}{s'(s'-s)}\,ds',$$

has to be treated numerically. First note that the integrand remains finite for $s' = s$, at least as long as $s \neq 4m_\pi^2$. The integral itself yields a finite value also in the latter

case. We therefore just have to calculate an ordinary integral.
Numerous algorithms were invented to perform numerical integrations. Many of them are quite sophisticated and show a very good convergence behaviour.
To get the Omnès function, one can make use of the fact that the occurring integrands are slowly varying functions and apply a rather trivial integration algorithm, as e.g. Simpson's formula (also referred to as Kepler's rule):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right).$$

This formula integrates cubic polynomials exactly.
It is a good idea to choose an algorithm which allows to subdivide the integration interval in such a way that what has already been calculated can be recycled. The Simpson formula has this feature. Performing one bisection of the interval $[a,b]$ the above integral may be approximated by:

$$\frac{1}{2}\frac{b-a}{6}\left(f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+3b}{4}\right)+f(b)\right).$$

Denoting the sum of the function values entering with weight 2 by Σ_2 and those with weight 4 by Σ_4, we may write:

$$\int_a^b f(x)\,dx \approx \Delta x\,(f(a)+2\Sigma_2+4\Sigma_4+f(b)),$$

where we start with $\Delta x=(b-a)/6$, $\Sigma_2=0$ and $\Sigma_4=f((a+b)/2)$. A bisection of the interval then amounts to $\Delta x\to\Delta x/2$, $\Sigma_2\to\Sigma_4$, whereas Σ_4 has to be recalculated. However, every function value once calculated is reused. The bisections are performed until the change in the approximation from one bisection to the next is less than a certain percentage of the result.
Another problem occurring when calculating the Omnès functions is the infinity of the upper integration limit. We get out of this difficulty by dividing up the interval $[4m_\pi^2,\infty)$ into intervals of finite length and adding up the results. Each of these integrals is calculated in the way discussed above. Clearly we can only compute a finite number of such contributions – but how many? First we proceed until we arrive at calculating an integral who's lower limit is bigger than s. If the upper limit is Λ, the absolute error from introducing this cutoff is less than $\frac{\delta(\infty)-\delta(\Lambda)}{\pi}\ln\left(\frac{\Lambda}{\Lambda-s}\right)$. We continue taking further intervals into account until the relative error due to cutting achieves the prescribed precision. The value of $\Omega(4m_\pi^2)$ may simply be replaced by $\frac{\Omega(4m_\pi^2-\epsilon)+\Omega(4m_\pi^2+\epsilon)}{2}$, where ϵ is numerically small.
In the case of complex s or real $s<4m_\pi^2$ we proceed in a similar way. The main difference is that no singularity of the integrand in equation (2.64) will be met.

3.2 Splines

In fact one could calculate the Omnès functions for any s. However, $|\Omega(s)|$ enters as a factor in the integrand of the integrals in (2.114). This forces us to know $\Omega(s)$ at extremely many points – each of them requiring the performance of a numerical integration taking relatively much time. The question of how to interpolate smooth functions thus arises at this point.
The probably most widely known interpolation method but not necessarily the best one is to introduce Lagrange polynomials. Much better suited to our problem

are splines. The original idea of splines is a physical one: if you take a true, elastic spline and bend it in such a way that it goes through some fixed points, it will take on the shape with minimal deformation energy. If the points are given by a not too rapidly varying function $f(x)$, the deformation energy per length is approximately proportional to $f''(x)^2$. Finding the configuration minimizing the deformation energy therefore means solving:

$$\delta E = \frac{\alpha}{2} \cdot \delta \int_a^b dx\, f''(x)^2 = 0,$$

where α denotes the constant of proportionality. In the following we assume, that $f(x)$ is known at the points $x_0 < x_1 < ... < x_n$ and denote $f(x_i)$ by y_i and $f''(x_i)$ by y_i''. Let $f(x) \to f(x) + \epsilon(x)$, where $\epsilon(x)$ has to vanish at the points $x = x_i$. For δE we get

$$\delta E = \alpha \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f''(x)\epsilon''(x)dx.$$

Performing two partial integrations leads to

$$\delta E = \alpha \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f^{(4)}(x)\epsilon(x)dx - \alpha \sum_{i=1}^{n} f''(x)\epsilon'(x)\Big|_{x_{i-1}}^{x_i} = 0.$$

This may be fulfilled independently of the special choice of ϵ, if we require

$$f^{(4)}(x) = 0\,, \quad \text{for } x \in [x_0, x_n]\,, \tag{3.2}$$
$$f''(x_0) = f''(x_n) = 0. \tag{3.3}$$

The first condition implies that $f(x)$ may be represented by a third order polynomial in each interval $[x_i, x_{i+1}]$. For $x \in [x_i, x_{i+1}]$ we write

$$f(x) = s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i\,. \tag{3.4}$$

Further requirements are that $f(x)$, $f'(x)$ as well as $f''(x)$ are continuous also at the points $x = x_i$. This special kind of splines defined uniquely by these conditions are called natural cubic splines. The smoothness requirements explicitly read:

$$\begin{aligned}
s_i(x_i) &= d_i = y_i\,,\\
s_i(x_{i+1}) &= a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + d_i = y_{i+1}\,,\\
s_i'(x_i) &= c_i\,,\\
s_i'(x_{i+1}) &= 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + c_i = c_{i+1}\,,\\
s_i''(x_i) &= y_i''(x_{i+1} - x_i)\,,\\
s_i''(x_i) &= 6a_i(x_{i+1} - x_i) + 2b_i = 6a_i(x_{i+1} - x_i) + y_i'' = y_{i+1}''\,.
\end{aligned}$$

The coefficients a_i, b_i follow from the last two of the above six equations:

$$\begin{aligned}
a_i &= \frac{y_{i+1}'' - y_i''}{6(x_{i+1} - x_i)}\,,\\
b_i &= \frac{y_i''}{2}\,.
\end{aligned}$$

Using the first two equations gives us:

$$\begin{aligned} c_i &= \frac{y_{i+1}-y_i}{x_{i+1}-x_i} - \frac{x_{i+1}-x_i}{6}(y''_{i+1}+2y''_i)\,, \\ d_i &= y_i\,. \end{aligned}$$

The only unknown quantities are the y'''s. The forth equation, however, connects the coefficients c_i and c_{i+1} – this leads to

$$h_{i+1}y''_{i+2} + 2(h_{i+1}+h_1)y''i + 1 + h_i y''_i = 6\left(\frac{y_{i+2}-yi+1}{h_{i+1}} - \frac{y_{i+1}-y_i}{h_i}\right)\,.$$

We end up with the task of solving a set of linear equations for the y'''s – we may combine them in a matrix notation $A\vec{y}'' = \vec{w}$:

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & & 0 \\ h_1 & 2(h_1+h_2) & & \\ & & \ddots & h_{n-2} \\ 0 & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix} \begin{pmatrix} y''_1 \\ y''_2 \\ \vdots \\ y''_{n-1} \end{pmatrix} =$$

$$6\begin{pmatrix} \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} - \frac{h_0 y''_0}{6} \\ \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{h_{n-2}} - \frac{h_{n-1}y''_n}{6} \end{pmatrix}\,. \qquad (3.5)$$

The matrix A is tridiagonal, which makes it especially easy to solve the linear system of equations (3.5). We may write A in the form $A = L \cdot R$, where

$$L = \begin{pmatrix} 1 & & & & 0 \\ l_1 & 1 & & & \\ & l_2 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & l_{n-2} & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} m_1 & r_1 & & & 0 \\ & m_2 & r_2 & & \\ & & m_3 & \ddots & \\ & & & \ddots & r_{n-2} \\ 0 & & & & m_{n-1} \end{pmatrix}.$$

For l_i, m_i and r_i we find:

$$\begin{aligned} r_i &= h_i\,, \\ m_1 &= 2(h_0+h_1)\,, \\ m_{i+1} &= 2(h_i+h_{i+1}) - h_i l_i\,, \\ l_i &= \frac{h_i}{m_i}\,. \end{aligned}$$

First the equation $L\vec{z} = \vec{w}$ is solved (forward substitution):

$$\begin{aligned} z_1 &= w_1\,, \\ z_i &= d_i - l_{i-1}z_{i-1}\,. \end{aligned}$$

Next the backward substitution is performed ($R\vec{y}'' = \vec{z}$):

$$\begin{aligned} y''_{n-1} &= \frac{z_{n-1}}{m_{n-1}}\,, \\ y''_i &= \frac{1}{m_i}(z_i - h_i y''_{i+1})\,. \end{aligned}$$

The fact that A is tridiagonal has the big advantage that finding the solution of the system of equations is only an $\mathcal{O}(n)$ problem. It is therefore possible to choose big n without getting unreasonably long computation times.
Here we will use splines to interpolate functions – clearly it would be too brute to use natural splines for this purpose. Instead of that we will use approximate values for y_0'' and y_n''.

3.3 Numerical calculation of the dispersion integrals

The integral equations (2.114) are solved iteratively. The amplitudes $M_I(s)$ are initialized with start values. From this one calculates the angular averages and then performs the integrals (2.114). This procedure is repeated until the results don't change more than a certain percentage from one iteration step to the next. First we have to rewrite the expression (2.99) for the angular averages to make it easy to describe the deformation of the integration path. The variable transformation $z = \frac{2}{\kappa(s)}\left(s' - \frac{3}{2}s_0 + \frac{1}{2}s\right) = \frac{2}{\kappa(s)}(s' - \sigma)$ leads to:

$$\hat{M}_0(s) = \frac{1}{\kappa(s)} \int\limits_C ds' \left(\frac{2}{3} M_0(s') + \frac{20}{9} M_2(s') + 2(s - s_0) M_1(s') + \frac{4}{3}(s' - \sigma) M_1(s') \right), \quad (3.6)$$

$$\hat{M}_1(s) = \frac{1}{\kappa^3(s)} \int\limits_C ds' \Big(6(s' - \sigma) M_0(s') - 10(s' - \sigma) M_2(s') + 9(s - s_0)(s' - \sigma) M_1(s') + 6(s' - \sigma)^2 M_1(s') \Big), \quad (3.7)$$

$$\hat{M}_2(s) = \frac{1}{\kappa(s)} \int\limits_C ds' \left(M_0(s') + \frac{1}{3} M_2(s') - (s' - \sigma) M_1(s') - \frac{3}{2}(s - s_0) M_1(s') \right). \quad (3.8)$$

The integration paths C are described in section (2.12).
Here we may take advantage of the spline representation of the amplitudes $M_I(s)$ – the integrands in the above equations are given by polynomials. Thus the integrals themselves may be solved exactly, the approximation is contained in the representation of the functions $M_I(s)$. We just have to divide the integration contour according to the points chosen to specify the splines.
The integral equations (2.114) are of the form

$$M_I(s) = \Omega_I(s) \left(P_I(s) + \int\limits_{4m_\pi^2}^{\infty} ds' \frac{F_I(s')}{\kappa(s')^{1+2n}(s' - s - i\epsilon)} \right), \quad (3.9)$$

where

$$F_I(s) = \frac{\sin\delta_I(s) \hat{M}_I(s)}{|\Omega_I(s)| {s'}^m}.$$

For $I = 0$ and $I = 2$ we have $m = 2$, $n = 0$ and for $I = 1$ $m = n = 1$. There is no problem with the functions $F_I(s)$ – but the zeros of $\kappa(s)$ need special attention, since they may cause the integrands in the integral equations to be singular.
At $s' = 4m_\pi^2$ the averages $\hat{M}_I$ are proportional to the path length and thus of $\mathcal{O}(|\kappa|)$. From the representation of the phase shifts (equation (2.126)) we see that, in the vicinity of $s' = 4m_\pi^2$, $\sin\delta_I(s')$ is of $\mathcal{O}(|\kappa|^{1+2n})$. The integrand in (3.9) therefore tends to zero as $s' \to 4m_\pi^2$.
The two other zeros of $\kappa(s)$ induce singularities of the integrand at the points

$$a = (m_\eta - m_\pi)^2 \text{ and } b = (m_\eta + m_\pi)^2.$$

We divide up the integrals into four parts ($a + g = b - h$):

$$\int_{4m_\pi^2}^{\infty} = \int_{4m_\pi^2}^{a-g} + \int_{a-g}^{a+g} + \int_{b-h}^{b+h} + \int_{b+h}^{\infty}. \tag{3.10}$$

The integration intervals have to be chosen in such a way that s does not lie on their boundaries except for $s = 4m_\pi^2$. For further use write $\kappa(s) = A(s) \cdot B(s) \cdot C(s)$, with

$$A(s) = \sqrt{\frac{s - 4m_\pi^2}{s}}, \quad B(s) = \sqrt{a - s}, \quad C(s) = \sqrt{b - s}.$$

The first part can be evaluated numerically as it stands, whereas the second and third one require a more careful treatment – I will give here detailed solutions.
With $G(s) = F(s)/(A(s)C(s))$ the second part reads

$$\int_{a-g}^{a+g} ds' \frac{G(s)}{(a - s')^{1/2+n}(s' - s - i\epsilon)}. \tag{3.11}$$

Let us consider first the case $n = 0$. Assume that $s \notin [a - g, a + g]$. We may use the same trick as applied to calculate the Omnès functions:

$$\int_{a-g}^{a+g} ds' \frac{G(s')}{(a - s')^{1/2}(s' - s - i\epsilon)} =$$
$$\int_{a-g}^{a+g} ds' \frac{G(s') - G(a)}{(a - s')^{1/2}(s' - s - i\epsilon)} + G(a) \int_{a-g}^{a+g} \frac{ds'}{(a - s')^{1/2}(s' - s - i\epsilon)}. \tag{3.12}$$

The first integral has got a finite integrand and is solved numerically. For the second one, Q_0, an analytic solution exists. By introducing the variable $x = s' - a$ we get:

$$Q_0 = \int_{-g}^{g} \frac{dx}{(-x)^{1/2}(x + a - s)}, \quad (a - s) \notin [-g, g].$$

This and the following integrals are solved using

$$\int \frac{dx}{(-x)^{1/2}(x + a - s)} = \frac{2\sqrt{x}}{\sqrt{-x}} \frac{\arctan\sqrt{x/(s - a)}}{\sqrt{a - s}}. \tag{3.13}$$

The relation

$$\arctan(x) = \frac{i}{2} \ln\left(\frac{1 - ix}{1 + ix}\right)$$

allows us to write Q_0 in a nice way with real arguments of ln and arctan.

$$Q_0(s,a,g) = \begin{cases} -\frac{1}{\sqrt{a-s}}\left(2i\arctan\sqrt{\frac{g}{a-s}} + \ln\frac{\sqrt{a-s}-\sqrt{g}}{\sqrt{a-s}+\sqrt{g}}\right) & \text{for } s < a. \\ -\frac{1}{\sqrt{s-a}}\left(2\arctan\sqrt{\frac{g}{s-a}} + i\ln\frac{\sqrt{s-a}-\sqrt{g}}{\sqrt{s-a}+\sqrt{g}}\right) & \text{for } s > a. \end{cases} \tag{3.14}$$

If $s \in [a-g, a+g]$ we proceed in the same way and add and subtract $G(a)$ in the numerator of the integrand in equation (3.11). One contribution is $G(a)I_0(s,a,g)$, where

$$I_0 = \int_{-g}^{g} \frac{dx}{(-x)^{1/2}(x+a-s-i\epsilon)}, \quad (a-s) \in [-g, g].$$

The solution is given by

$$I_0(s,a,g) = \begin{cases} \frac{1}{\sqrt{a-s}}\left(-\ln\frac{\sqrt{g}-\sqrt{a-s}}{\sqrt{g}+\sqrt{a-s}} - 2i\arctan\sqrt{\frac{g}{a-s}} + i\pi\right) & \text{for } s < a. \\ \frac{1}{\sqrt{s-a}}\left(-i\ln\frac{\sqrt{g}-\sqrt{s-a}}{\sqrt{g}+\sqrt{s-a}} - 2\arctan\sqrt{\frac{g}{s-a}} + \pi\right) & \text{for } s > a. \end{cases} \tag{3.15}$$

In the limit $s \to a$ the integral I_0 remains finite. This is not true for the unphysical boundary $\mathrm{Im}(s)=-i\epsilon$ – the contributions from arctan and the residue don't cancel one another.
In the case $s \notin [a-g, a+g]$ the first integral on the right hand side of equation (3.12) could be done numerically. This is no longer true now, since the integrand suffers from a second singularity at $s' = s$, i.e. solving this integral requires calculating a Cauchy principal value. We have to represent this contribution as a sum of two integrals – one running from $a-g$ to $p = (s+a)/2$, the other one from p to $a+g$. One of them has got a finite integrand and can be integrated by a numerical method, whereas the other one, let us say the first one, has to be written as:

$$\int_{a-g}^{p} ds' \frac{G(s')-G(a)}{(a-s')^{1/2}(s'-s-i\epsilon)} = \int_{a-g}^{p} ds' \frac{G(s')-G(s)}{(a-s')^{1/2}(s'-s)} + (G(s)-G(a)) \int_{a-g}^{p} \frac{ds'}{(a-s')^{1/2}(s'-s-i\epsilon)}.$$

Again we have divided up an integral which could not be left to the computer as it stood into a contribution suitable for numerical analysis and a part which can be solved analytically – the second one which we write as $(G(s)-G(a))P_0(s,a,g)$. P_0 is the same as I_0 except for the integration limits which are $\{a-g, p\}$ for $s < a$ and $\{p, a+g\}$ for $s > a$.

$$P_0(s,a,g) = \begin{cases} \frac{1}{\sqrt{a-s}}\left(\ln\frac{\sqrt{a-s}-\sqrt{a-p}}{\sqrt{a-s}+\sqrt{a-p}} - \ln\frac{\sqrt{g}-\sqrt{a-s}}{\sqrt{g}+\sqrt{a-s}} + i\pi\right) & \text{for } s < a. \\ \frac{1}{\sqrt{s-a}}\left(i\ln\frac{\sqrt{s-a}-\sqrt{p-a}}{\sqrt{s-a}+\sqrt{p-a}} - i\ln\frac{\sqrt{g}-\sqrt{s-a}}{\sqrt{g}+\sqrt{s-a}} + \pi\right) & \text{for } s > a. \end{cases} \tag{3.16}$$

It would have been possible to treat the principal value integrals induced by $i\epsilon$ by choosing a numerical ϵ and solving all integrals except I_0 numerically. However, for my taste this method is ambiguous: if you choose ϵ too large, the result will clearly tend to be off from the real value by a rather large amount and a too small ϵ might cause problems with errors coming from the limited precision of computer numbers.

Thus one would have to search the "correct" ϵ – and I don't like procedures like that, especially because $s = a$ is exactly on the boundary of the physical region. For $n = 1$ the procedure is the same. In the numerator of the integrand of

$$\int_{a-g}^{a+g} ds' \frac{G(s)}{(a-s')^{3/2}(s'-s-i\epsilon)}$$

we add and subtract $G(a) + (s'-a)G'(a)$, where now $G(s) = F(s)/(A(s)C(s))^3$. We are left with ordinary integrals and integrals of the type

$$\int_{-g}^{g} \frac{dx}{(-x)^{3/2}(x+a-s-i\epsilon)} .$$

This time the singularity at $x = s' - a = 0$ is so strong that the integration contour needs to be deformed:

$$\int_{-g}^{g} = \int_{-g}^{-\delta} + \int_{C} + \int_{\delta}^{g},$$

where C is a semi circle of radius δ. The path C has to lie in the other half-plane than the singularity, namely in the lower one. In virtue of

$$\int \frac{dx}{(-x)^{3/2}(x+a-s-i\epsilon)} = \frac{-2}{(a-s)(-x)^{3/2}} \left(x + \frac{x^{3/2}}{(a-s)^{1/2}} \arctan \sqrt{\frac{x}{a-s}} \right)$$

we find the functions Q_1, I_1 and P_1 corresponding to Q_0, I_0 and P_0.
For $s < a$ we have:

$$\begin{aligned} Q_1(s,a,g) &= -\frac{2+2i}{(a-s)\sqrt{g}} + \frac{Q_0(s,a,g)}{a-s} , \\ I_1(s,a,g) &= -\frac{2+2i}{(a-s)\sqrt{g}} + \frac{I_0(s,a,g)}{a-s} , \\ P_1(s,a,g) &= \frac{-2}{a-s} \left(\frac{1}{\sqrt{g}} - \frac{1}{\sqrt{a-p}} \right) + \frac{P_0(s,a,g)}{a-s} . \end{aligned} \tag{3.17}$$

and for $s > a$ we get:

$$\begin{aligned} Q_1(s,a,g) &= \frac{2+2i}{(s-a)\sqrt{g}} - \frac{Q_0(s,a,g)}{s-a} , \\ I_1(s,a,g) &= \frac{2+2i}{(s-a)\sqrt{g}} + \frac{I_0(s,a,g)}{s-a} , \\ P_1(s,a,g) &= \frac{2i}{s-a} \left(\frac{1}{\sqrt{g}} - \frac{1}{\sqrt{p-a}} \right) - \frac{P_0(s,a,g)}{s-a} . \end{aligned} \tag{3.18}$$

With the formulas presented here it is now an easy task to put the pieces together. E.g for $s \in (a-g, a)$ we get:

$$\begin{aligned} &\int_{a-g}^{a+g} ds' \frac{G(s')}{(a-s')^{3/2}(s'-s-i\epsilon)} = \\ &\int_{a-g}^{p} ds' \frac{G(s') - G(s)}{(a-s')^{3/2}(s'-s)} + \int_{p}^{a+g} ds' \frac{G(s') - G(a) - (s'-a)G'(a)}{(a-s')^{3/2}(s'-s)} + \\ &G(a)I_1 - G'(a) \left(I_0 + \frac{2}{\sqrt{g}} - \frac{2}{\sqrt{a-p}} \right) + \big(G(s) - G(a) - (s-a)G'(a) \big) P_1 . \end{aligned}$$

In the case $s = a$ the above equation would require an even more detailed treatment to get it in a form suitable for computers. However, one may solve the problem by avoiding $s = a$. It suffices to note that the result will stay finite also in this case. Defining $H_I = \frac{\sin\delta_I \hat{M}_I}{\kappa^{1+2n}|\Omega_I|}$, the fourth part of equation (3.10) reads

$$\int\limits_{b+h}^{\infty} ds' \frac{H_I(s')}{s'^m(s'-s-i\epsilon)} .$$

For $s < b+h$ the integrand causes no problem. Otherwise we write

$$\int\limits_{b+h}^{\infty} ds' \frac{H_I(s')}{s'^m(s'-s-i\epsilon)} = \int\limits_{b+h}^{\infty} ds' \frac{H_I(s')-H_I(s)}{s'^m(s'-s)} + H_I(s) \int\limits_{b+h}^{\infty} \frac{ds'}{s'^m(s'-s-i\epsilon)} .$$

The second integral yields

$$\int\limits_{b+h}^{\infty} \frac{ds'}{s'^m(s-s'-i\epsilon)} = \begin{cases} i\pi - \frac{1}{(b+h)s} - \frac{1}{s^2}\ln\frac{s-b-h}{b+h} & \text{for } I = 0,2. \\ \frac{i\pi}{s} + \frac{1}{s}\ln\frac{b+h}{s-b-h} & \text{for } I = 1. \end{cases}$$

In either case we have to perform a numerical integration with an infinite upper limit. The most easy way to overcome this difficulty is to introduce a cutoff Λ. This procedure is legitimate since e.g. the behaviour of the phase shifts at energies greater than about 1 GeV is subject to considerable phenomenological uncertainties. Of course one has to verify that Λ is big enough not to influence the final result.
To get the expressions for I_n and P_n at the lower rim, one just has to change the signs of the terms proportional to π and $i\pi$ in equation (3.15) and (3.16).

3.4 How to represent Ω_I, M_I and $\hat{M}_I$

For the numerical evaluation of (2.114) I have exclusively used splines to interpolate $|\Omega_I|$, M_I and $\hat{M}_I$. So let's determine in which region we have to represent these functions.
First it is a good idea to calculate Ω_I at every point we choose to be a supporting point for the amplitudes M_I. This way we won't need splines to interpolate the Omnès functions themselves but only for representing $|\Omega_I|$.
In principle we need to know M_I only inside the physical region, namely in the interval extending from $4\,m_\pi^2$ up to $(m_\eta - m_\pi)^2 \approx 9.3\,m_\pi^2$. However, performing an iteration step requires to supply information on the amplitude at additional points in order to calculate the angular averages $\hat{M}_I$. The angular averages have to be evaluated for real $s \in [4m_\pi^2, \Lambda]$, where Λ denotes the cutoff introduced in the last section. We will choose Λ certainly not smaller than about $100\,m_\pi^2$ and will therefore meet values of s for which the path described in section (2.12) has got complex endpoints. Comparing with figures 2.3(a) - 2.3(d), one sees that we thus have to calculate M_I in a rectangle containing the complex region $[-3m_\pi^2, 5m_\pi^2] \times [-4m_\pi^2\, i, 4m_\pi^2\, i]$. Choosing another shape than a rectangle would only unnecessary complicate the program. For $s > 30\,m_\pi^2$ we have $s_+ \gtrsim -s$ and $s_- \lesssim 0$. This implies that M_I is needed on the real axis on an interval with a lower limit equal to $-\Lambda$. Although it would be sufficient to set the upper limit somewhat higher than $(m_\eta - m_\pi)^2$ (in order to get an interpolation of higher accuracy), it was chosen even higher. Calculating M_I up to $s > (m_\eta + m_\pi)^2$ may serve as a check on the correct handling of the singularities generated by κ. If e.g. the integrands in (2.114) are represented not precisely enough, the results for M_I will show discontinuities

in the vicinity of the zeros of κ. These discontinuities are much more distinct at $s = (m_\eta + m_\pi)^2$ than at $s = (m_\eta - m_\pi)^2$.
For complex functions real- and imaginary parts are represented separately, i.e. each by a spline of its own. Inside the rectangle described above, splines are put parallel to the real axis. From them one may calculate function values that are used to get a spline parallel to the imaginary axis.
Even though splines have many nice properties one must not use them blindly. The functions M_I are expected to have a cusp at $s = 4m_\pi^2$ (compare with the one loop result). In order not to smear this cusp we have to split up the interpolating interval into a piece left to $4m_\pi^2$ and one right to this point. This way it is no problem to get the cusp right. For $\hat{M}_I$ we have even two critical points: $s = (m_\eta \pm m_\pi)^2$. At these points the real- and imaginary parts are equipped with a cusp. It isn't only crucial to split the splines there – one also has to set the points very dense in this region, otherwise the check mentioned above is determined to fail.
The whole iteration procedure has to be performed for different Λ. Also the point densities have to be varied in order to find a minimal number of points such that a further increase does not change the result. Doing so, I found that a save result is achieved with $\Lambda = 200\, m_\pi^2$. For the results quoted in this work, M_I was calculated up to $200\, m_\pi^2$ and the rectangle I set to $[-6m_\pi^2, 12m_\pi^2] \times [-6m_\pi^2\, i, 6m_\pi^2\, i]$.
Within the rectangle the amplitudes have to be known on the upper as well as on the lower rim of the cut. We have to keep in mind that on the lower rim only points with $s < (m_\eta - m_\pi)^2$ should be taken into account, otherwise we would include the singularity of the amplitude at $s = (m_\eta - m_\pi)^2 - i\epsilon$. Inside the rectangle the functions $M_I(s)$ are calculated at the points $s = \frac{m_\pi^2}{16} \cdot (m, n\, i)$ with $m,\ n \in \mathbb{Z}$. Outside a lower point density is sufficient. Besides that, it is favourable to add some extra points: at $\mathrm{Re}(s) = s_A$ and in the vicinity of $\mathrm{Re}(s) = 4m_\pi^2$. $\hat{M}_I$ is equipped with a much higher point density: up to 256 points per m_π^2 in the vicinity of $s = (m_\eta \pm m_\pi)^2$ – maybe a little bit of luxury.

3.5 Remark on computer generated difficulties

It is a known fact that many lines of a computer program are written twice before the computer gives the expected output. The source of possible errors is almost arbitrarily high. Often it is only necessary to compare the output on the screen with what we wanted to get – if this is o.k., we know the program is working as it should.
However, when solving problems as our integral equations we have no possibility to check the results directly. In this case we have to find a method to test the principle of the implemented algorithm. But before this can be done, we clearly have to check very carefully the whole code in order to be sure to have implemented what we intended to. Every information we have on the properties the result should show is a great help to get on the track of software bugs. If we e.g. expect the output to be a smooth function and get instead a wildly oscillating one, this will force us automatically to check the corresponding routines. But even if the first result seems to be trustworthy, it is absolutely inevitable to check every procedure separately and to look at intermediate results of the calculation. Many bugs are found where one never imagined there was a possible source of error.
A frequent error is that a variable is not or wrongly initialized. It may take very much time to find such an error. Another problem which caused me searching bugs for days: one easily forgets that the computer represents numbers only with a limited precision. This is illustrated by the following example. Let us assume that we have to calculate $f(x)/g(x)$ at several points (e.g. in the course of a numerical integration). If this quantity has to be evaluated at the point x_0 with $f(x_0) =$

$g(x_0) = 0$, the result has to be given explicitly. It may happen that, because of the limited precision, the computer tries to calculate $f(x)/g(x)$ not exactly at $x = x_0$ (as expected) but only very close to – and in this case the computer is not able to return a result of sufficient accuracy. The result may be off by a factor 10^6 or more – the numerical integration will never yield the correct answer. To circumvent this problem we may e.g. set $f(x)/g(x) = f(x_0)/g(x_0)$ in an entire vicinity of x_0 that has to be much bigger than the computer ϵ (the difference between 1 and the smallest value greater than 1 that is representable).

4 Numerical results

4.1 How to get the decay rates from the transition amplitude

Before I come to present the numerical results of the dispersive analysis, let me add some words concerning the connection between transition amplitude and decay rate.
The transition amplitude as such is not very interesting, unless we may use it to make predictions of quantities measured in experiments, as e.g. cross sections and transition rates. The transition amplitude for a process $\alpha \to \beta$ is given by the scattering-matrix element $S_{\beta\alpha}$. In the case where $\alpha \neq \beta$ we have

$$S_{\beta\alpha} = i(2\pi)^4 \delta^4(p_\beta - p_\alpha) T_{\beta\alpha}.$$

The $S-$ and $T-$matrix are defined in subsection 2.8.1. The transition probability is $|S_{\beta\alpha}|^2$ and thus proportional to the square of a Dirac δ-function, ensuring energy-momentum conservation and making one not too happy at the first glance.
To get the right answer, remember that we can observe merely wave packets and not incoming particles in a momentum eigenstate, an idealization we have used so far. The wave packets are, however, peaked quite sharply around given momenta.
In our case we write for the incoming η-particle

$$|\eta\rangle = \int d\tilde{k}\, f(k) |\eta(k)\rangle\,, \qquad \text{where} \qquad d\tilde{k} = \frac{d^3k}{(2\pi)^3 2k^0}\,.$$

The transition probability for $|\eta\rangle \to |\mathit{final}\rangle$ is then given by

$$w = (2\pi)^8 \int d\tilde{k}\, d\tilde{q}\, \delta^4\big(P_f - k\big)\, \delta^4\big(P_f - q\big)\, f(k) f^\star(q)\, A\big(k, p_f\big) A^\star\big(q, p_f\big)\,.$$

Here P_f stands for the total four-momentum of the final state and $A(k, p_f)$ denotes the invariant amplitude for the decay $\eta \to \mathit{final\ state}$ where p_f collects the dependence of A on the three momenta of the particles occurring in the final state. The second δ-function may be replaced by $\delta^4(k-q)$ – we can do $\int d^3q$ and have $q^0 = k^0$. The factor $\delta(k^0 - q^0)$ is represented as $(2\pi)^{-4} \int dt \exp[i(k^0 - p^0)t]$.

$$w = (2\pi)^4 \int \frac{d\tilde{k}\, dt}{(2\pi)^3 2k^0}\, \delta\big(k^0 - q^0\big)\, \delta^4\big(P_f - k\big)\, \big|f(k)\big|^2\, \big|A\big(k, p_f\big)\big|^2\,.$$

Since $f(k)$ is peaked around p_η, k may be approximated by p_η, except for the k appearing in $f(k)$. The normalization condition set upon the state $|\eta\rangle$, namely $\langle\eta|\eta\rangle = 1$, implies $\int d\tilde{k}\, |f(k)|^2 = 1$. The quantity dw/dt gives us the decay rate for a decay into a specific state.
The differential decay rate for decays into final states characterized by momenta $\vec{p}_i$

in ranges d^3p_i is:

$$d\Gamma = \frac{(2\pi)^4}{2p_\eta^0}\,\delta^4\big(p_1+p_2+p_3-p_\eta\big)\big|A(s,t,u)\big|^2\,d\tilde{p}_1\,d\tilde{p}_2\,d\tilde{p}_3\,.$$

To get Γ we go over to the rest frame of the η and proceed in the standard way:

$$\delta^4\big(p_1+p_2+p_3-p_\eta\big) = \delta\big(p_1^0+p_2^0+p_3^0-m_\eta\big)\,\delta^3\big(\vec{p}_1+\vec{p}_2+\vec{p}_3\big).$$

The δ^3-function cancels $\int d^3p_1$ and implies $\vec{p}_1 = -\vec{p}_2-\vec{p}_3$.
Next we go over to polar coordinates:

$$d^3p_2\,d^3p_3 = |p_2||p_3|p_2^0p_3^0\;dp_2^0\;dp_3^0\;d\Omega_3\;d\varphi_{23}\;d\cos\vartheta_{23}.$$

Note that $\delta\big(p_1^0+p_2^0+p_3^0-m_\eta\big)$ is of the form $\delta\big(f(\cos\vartheta_{23})\big)$ – the relation

$$p_1^0 = \big(m_\pi^2+|p_2|^2+|p_3|^2+2|p_2||p_3|\cos\vartheta_{23}\big)^{1/2}$$

together with

$$\frac{\partial}{\partial\cos\vartheta_{23}}\big(p_1^0+p_2^0+p_3^0-m_\eta\big) = \frac{|p_2||p_3|}{p_1^0}$$

allows us to write it as

$$\delta\big(p_1^0+p_2^0+p_3^0-m_\eta\big) = \delta\big(\cos\vartheta_{23}-\cos\vartheta_0\big)\frac{p_1^0}{|p_2||p_3|}\,.$$

$\cos\vartheta_0$ is the zero of the argument of $f(x)$.
Putting these pieces together, we get:

$$\Gamma = \frac{1}{16m_\eta(2\pi)^5}\int dp_2^0\,dp_3^0\,d\Omega_3\,d\varphi_{23}\,\big|A(s,t,u)\big|^2\,.$$

In the rest frame of the η the relations between the energies of the pions and the Mandelstam variables are of a simple form. In the case of $\eta\to\pi^+\pi^-\pi^0$ we have:

$$p_1^0 = \frac{m_\eta^2+m_{\pi^+}^2-t}{2m_\eta}\,,\quad p_2^0 = \frac{m_\eta^2+m_{\pi^+}^2-u}{2m_\eta}\quad\text{and}\quad p_3^0 = \frac{m_\eta^2+m_{\pi^0}^2-s}{2m_\eta}\,. \tag{4.1}$$

So we may do the angular integrals by changing the integration variables. We are left with

$$\Gamma = \frac{1}{256\pi^3m_\eta^3}\int ds\,du\,\big|A(s,t,u)\big|^2\,. \tag{4.2}$$

It remains to determine the phase space to integrate over. It is fully fixed by energy-momentum conservation. The difference between the masses m_{π^+} and m_{π^0} has quite a large influence on the relevant phase space and may not be neglected. The relation $s+t+u = 2m_{\pi^+}^2+m_{\pi^0}^2+m_\eta^2 = 3s_0$ is equivalent to $p_1^0+p_2^0+p_3^0 = m_\eta$. In the rest frame of the η, momentum conservation reads $\vec{p}_1+\vec{p}_2+\vec{p}_3 = 0$. If ϑ denotes the angle between $\vec{p}_1$ and $\vec{p}_3$, we may write

$$|\vec{p}_2|^2 = |\vec{p}_1|^2+|\vec{p}_3|^2-2\cos\vartheta|\vec{p}_1||\vec{p}_3|.$$

Using the expressions (4.1) allows us to express $\cos\vartheta$ in terms of the Mandelstam variables s and u: $\cos\vartheta = f(s,u)$. Momentum conservation now reduces to a constraint on the function $f(s,u)$: $-1\le f(s,u)\le 1$. From this we find the relations expressing the integration limits of the u-integration by s:

$$u_\pm(s) = \frac{1}{2}\left(3\,s_0-s\pm\frac{\sqrt{m_{\pi^0}^4+\big(s-m_\eta^2\big)^2-2m_{\pi^0}^2\,\big(s+m_\eta^2\big)}\sqrt{s-4\,m_{\pi^+}^2}}{\sqrt{s}}\right).$$

The integration variable s in (4.2) runs from s_{min} to s_{max}, given by

$$s_{min} = 4\,m_{\pi^+}^2 \qquad \text{and} \qquad s_{max} = (m_\eta - m_{\pi^0})^2\,.$$

We may finally write down (4.2) with specified integration limits:

$$\Gamma = \frac{1}{256\pi^3 m_\eta^3} \int_{s_{min}}^{s_{max}} ds \int_{u_-(s)}^{u_+(s)} du \left|A(s,t,u)\right|^2 . \tag{4.3}$$

For the neutral decay mode, $\eta \to 3\pi^0$, we have to replace m_{π^+} by m_{π^0}. Besides that the above formula for the decay rate has to be multiplied by an additional factor $1/3!$, accounting for the indistinguishability of the three outgoing π^0-particles.
The relevant phase space for the decay $\eta \to \pi^+\pi^-\pi^0$ is depicted in the following figure.

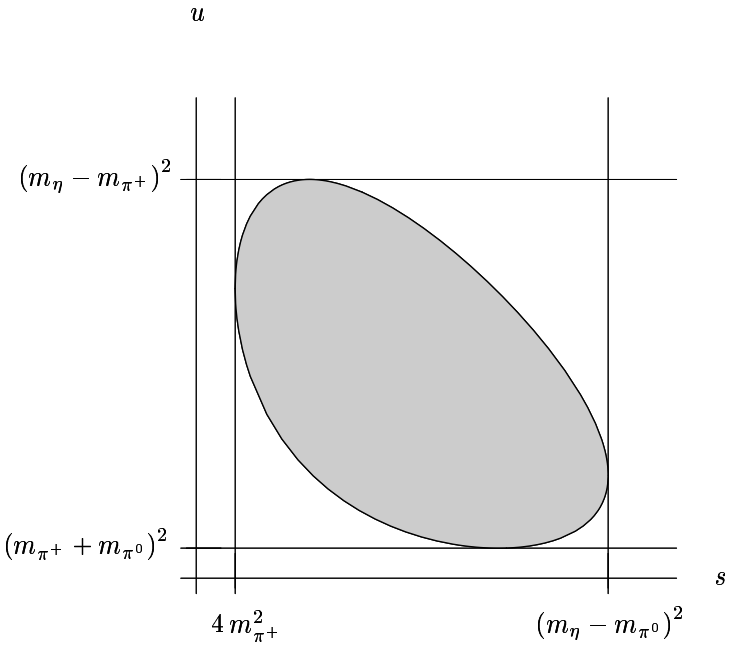


Figure 4.1: Relevant phase space for the decay $\eta \to \pi^+\pi^-\pi^0$.

4.2 Numerical results for M_I

Solving the integral equations (2.114) as described in the last chapter, the result for the functions $M_I(s)$ and for the amplitude $M(s,t,u)$ along the line $s = u$ looks as shown in figure (4.2). The plots correspond to $\beta_1 = 6.5\,\mathrm{GeV}^{-4}$, $\gamma_0 = 0$ and phase shifts according to set II.

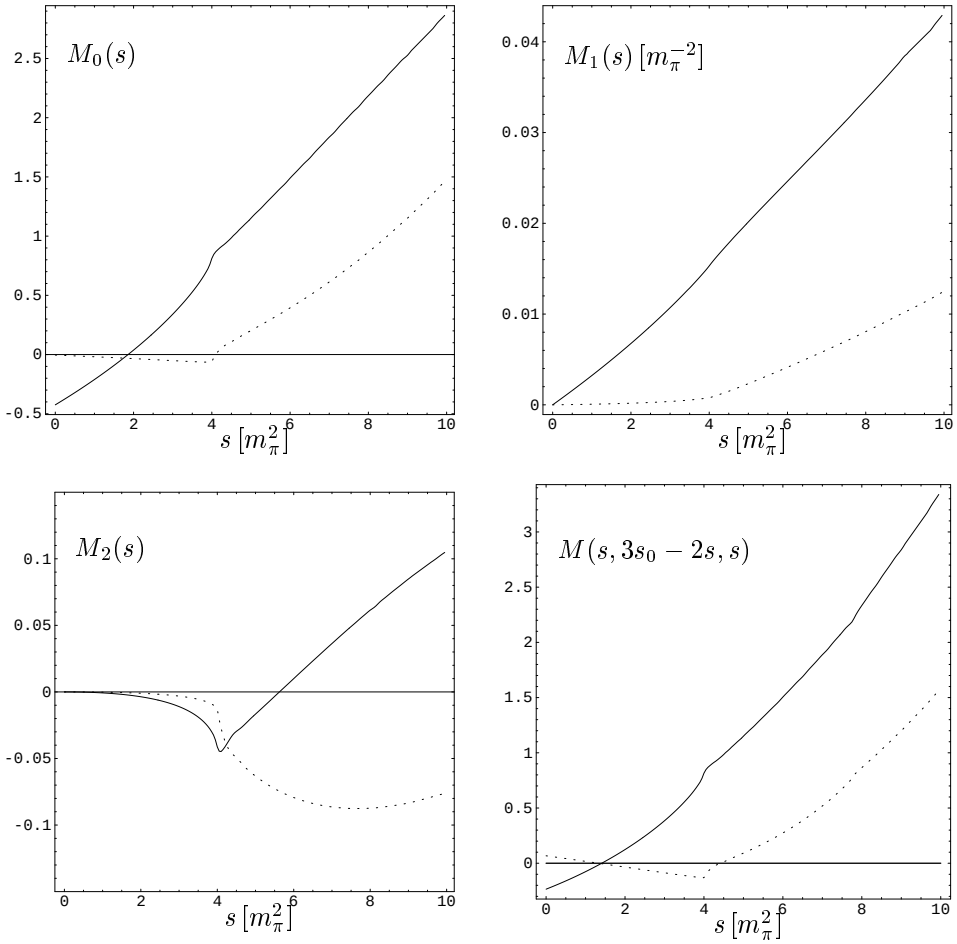


Figure 4.2: The function $M_I(s)$ and the amplitude $M(s,t,u)$ along the line $s = u$. The full lines represent the real parts and the dashed lines the imaginary parts.

4.3 Branching ratio $\Gamma_{3\pi^0}/\Gamma_{\pi^+\pi^-\pi^0}$

The current algebra prediction for the branching ratio $r = \Gamma_{3\pi^0}/\Gamma_{\pi^+\pi^-\pi^0}$ is

$$r = 1.51\,,$$

whereas the one loop calculation (ref. [7]) leads to

$$r = 1.43\,.$$

This result is only slightly affected when taking electromagnetic corrections into account. The numerical value given in ref. [8] is

$$1.42 \leq r \leq 1.43\,.$$

The values quoted by the Particle Data Group are

$$\begin{aligned} r &= 1.39 \pm 0.05 \quad \text{fitted value,} \\ r &= 1.34 \pm 0.10 \quad \text{average value.} \end{aligned}$$

The dispersive analysis of $\eta \to 3\pi$ leads to

$$\begin{aligned} r &= 1.44 \pm 0.01 \quad \text{set A.} \\ r &= 1.43 \pm 0.01 \quad \text{set I,} \\ r &= 1.43 \pm 0.01 \quad \text{set II.} \end{aligned}$$

The error is due to the uncertainties associated with γ_0 and β_1. The modified integral equations yield the same prediction for r.
The results obtained by means of dispersion relations coincide with those from the one-loop calculations. They are all somewhat higher than the fitted value of the PDG, but do agree with it within the error bar.

4.4 Dalitz plot distribution

In order to analyze the Dalitz plot distribution we introduce the standard coordinates x and y, defined by

$$\begin{aligned} x &= \frac{\sqrt{3}}{2m_\eta \Delta}(u-t)\,, \\ y &= \frac{3}{2m_\eta \Delta}\left((m_\eta - m_{\pi^0})^2 - s\right) - 1\,, \\ \Delta &= m_\eta - 2m_{\pi^+} - m_{\pi^0}\,. \end{aligned} \tag{4.4}$$

The Dalitz plot distribution may be approximated by a second order polynomial

$$P(x,y) = N(1 + ay + by^2 + cx^2).$$

The amplitude $M(s,t,u)$ is symmetric in t and u. Therefore no odd power of x occurs in P. The coefficients a, b, c are determined by minimizing the integral

$$\int\limits_{\substack{\text{physical}\\ \text{region}}} dx\,dy\,\left(|A(s,t,u)|^2 - P(x,y)\right)^2$$

With $\beta_1 = 6.5\,\text{GeV}^{-4}$ and $\gamma_0 = 0$ we get the following values for a, b and c:

Set	a	b	c
I	-1.28	0.44	0.061
II	-1.28	0.45	0.059
A	-1.16	0.29	0.067

We compare the values of a, b and c obtained from the dispersive analysis with those predicted by the one loop calculation and those from fitting experimental data. The values from the dispersive analysis are based on set II. The error is due to the uncertainties of β_1 and γ_0 and is merely an estimate.

Method / Source	a	b	c
Dispersive analysis	-1.30 ± 0.03	0.45 ± 0.05	0.06 ± 0.01
One loop result	-1.33	0.42	0.08
Layter et al.	-1.08 ± 0.014	0.034 ± 0.027	0.046 ± 0.031
Gormley et al.	-1.17 ± 0.02	0.21 ± 0.03	0.06 ± 0.04

4.5 Final remarks

We have investigated the decay $\eta \to 3\pi$ by means of dispersion relation technique. This allowed us, in principle, to determine the quark mass ratio $Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ to rather high accuracy.
To conclude let me shortly discuss the uncertainties associated with the dispersive analysis of $\eta \to 3\pi$ presented in this work.

- Determining the subtraction constants β_1 and γ_0 by using the one-loop result gives not absolutely sharp values. The numerical analysis shows that the shifts in β_1 and γ_0 due to contributions from above $K\bar{K}$-threshold have quite a large effect on the prediction of Q.
  The requirement that the dispersive result for the amplitude coincides with the one-loop prediction at the Adler zero s_A relies on the assumption that the χPT series converges rapidly in this region – the position of s_A is protected by $SU(2) \times SU(2)$ symmetry, whereas the slope is not.

- The phase shifts have quite a large effect on the value of Q. This is illustrated by the large difference between the prediction for Q obtained with set A and sets I and II, respectively. However, the results corresponding to set I and II are very close to each other and are supposed to give the better description of the decay amplitude than set A. Nevertheless, definite knowledge of the phase shifts is desirable.

- We have seen that taking into account the inelasticity in the $\pi\pi$ final state interactions changes the result for Q only slightly. The attempt we made to account for the inelasticity effects, is not the best one can think off – it gives rather a hint that a precise analysis of inelastic contributions should not yield a surprisingly different result. As a next step, a more detailed analysis of the contributions from the region above $K\bar{K}$-threshold might be performed. This would yield additional discontinuities, involving further subtraction constants to be determined.

- The experimental uncertainties in the decay rate are quite large. More precise measurement of the rate would allow a more accurate determination of the quark mass ratio Q.

Disregarding the result obtained with set A, I estimate the error due to theoretical uncertainties in the decay amplitude at 10%, what leads to an error of 2.5% in the prediction for Q. Adding quadratically the theoretical and experimental error, we get $Q = 22.8 \pm 0.8$, based on set II.

Appendix

A.1 Feynman propagator

To start let me, for convenience, write down the Euler integral representation of the Γ-function and quote a relation widely used in this and the following section:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt\,, \quad \Gamma(z+1) = z \cdot \Gamma(z).$$

The Feynman propagator $\Delta(x)$ obeys the differential equation

$$\left(\Box + m^2\right) \Delta(x) = \delta^d(x). \tag{A.1}$$

With the help of a Fourier transformation one finds the solution to this equation:

$$\Delta(x) = \frac{1}{(2\pi)^d} \int d^dk\, \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon}\,. \tag{A.2}$$

The $i\epsilon$ prescription removes the singularity of the integrand at $k^2 = m^2$.
The quantity $\Delta(0)$ appearing in one-loop calculations is ultraviolet divergent if we set $d = 4$. Performing a Wick rotation ($x^0 \to ix^d$), we get

$$\Delta(0) = \frac{i}{(2\pi)^d} \int \frac{d^dk}{m^2 + k^2}\,,$$

where now k^2 is the square of the euclidean norm of k. Introducing an integral representation for $(m^2 + k^2)^{-1}$ leads to

$$\Delta(0) = \frac{i}{(2\pi)^d} \int_0^1 d\lambda \int d^dk\, e^{-\lambda(m^2+k^2)} = i \frac{m^{d-2}}{(4\pi)^{d/2}} \int_0^1 d\lambda\, \lambda^{\frac{d}{2}} e^{-\lambda} = i \frac{m^{d-2}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right).$$

This representation of $\Delta(0)$ is defined also for non-integer values of d. We may expand the Γ-function around $d = 4$:

$$\Gamma\left(1 - \frac{d}{2}\right) = \frac{2}{d-4} - \Gamma'(1) - 1 + \mathcal{O}(d-4)\,.$$

The singularity turns out to be a simple pole at $d = 4$.
The quantity $\Delta(0) - im^2(4\pi)^{-2}/(d-4)$ remains finite as $d \to 4$. The two terms are of different dimension and the difference thus depends on the units used for the mass. This dependence may be shifted onto an arbitrary renormalization scale μ of dimension mass. The renormalized propagator at the origin,

$$\Delta^r(0) = \Delta(0) - 2im^2\lambda, \tag{A.3}$$

yields a finite value as $d \to 4$, where λ is defined by

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\Big(\ln 4\pi + \Gamma'(1) + 1\Big)\right\}. \tag{A.4}$$

We have identified the singular part according to the modified minimal subtraction scheme. The advantage thereof is, that the result for the finite part won't contain ugly contributions proportional to $\ln 4\pi$ and $\Gamma'(1)$.
In the limit $d \to 4$ the result for $\Delta^r(0)$ reads:

$$\Delta^r(0) = \frac{im^2}{16\pi^2}\ln\frac{m^2}{\mu^2}. \tag{A.5}$$

A.2 One-loop integrals

In this appendix I will discuss the one-loop integrals occurring in section (2.8.3) and give some useful relations. For short we will write

$$\frac{d^dk}{(2\pi)^d} = d\tilde{k}.$$

Performing a Wick rotation and making use of the integral representation

$$\frac{(q^2)^\beta}{(m^2+q^2)^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1}\, e^{-tm^2}\,(-\partial/\partial t)^\beta\, e^{-tq^2},$$

we readily check the formula

$$\int d\tilde{q}\frac{(q^2)^\beta}{(m^2-q^2-i\epsilon)^\alpha} = \frac{(-1)^\beta i}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\beta-d/2)\,\Gamma(\beta+d/2)}{\Gamma(\alpha)\,\Gamma(d/2)}\,m^{d+2(\beta-\alpha)} \tag{A.6}$$

for integer values of β.

- Let's first consider the one-loop function $J(s)$ $(s = p^2)$ defined by

$$J(s) = -i\int\frac{d\tilde{q}}{\big(m^2-q^2\big)\big(m^2-(p-q)^2\big)}. \tag{A.7}$$

  To evaluate $J(s)$ one uses the Feynman-Schwinger trick:

$$\frac{1}{AB} = \int_0^1\frac{dx}{\big((1-x)A+xB\big)^2}.$$

  This yields:

$$J(s) = -i\int d\tilde{q}\int_0^1\frac{dx}{\big\{(1-x)\big(m^2-q^2\big)+x\big(m^2-p^2+2pq-q^2\big)\big\}^2} =$$

$$-i\int d\tilde{q}\int_0^1\frac{dx}{\big(m^2-q^2-xp^2+2xpq\big)^2} = -i\int_0^1 dx\int\frac{d\tilde{q}}{\big(m^2-q^2+x(x-1)p^2\big)^2}.$$

  If we now invoke equation (A.6) and expand the result around $d = 4$, we find

$$J(s) = \frac{1}{16\pi^2}\left(\frac{-2}{d-4} + \ln 4\pi + \Gamma'(1) + \mathcal{O}(d-4)\right)\int_0^1 dx\, g(x,s)^{d-4},$$

where $g^2(x,s) = m^2 + x(x-1)s$.
The pole of $J(s)$ is contained in $J(0)$. The quantity $\bar{J}(s) = J(s) - J(0)$ is finite in the limit $d \to 4$:

$$\bar{J}(s) = -\frac{1}{16\pi^2}\int_0^1 dx \ln\frac{g^2(x,s)}{g^2(x,0)} = \frac{1}{16\pi^2}\left\{\sigma \ln\frac{\sigma-1}{\sigma+1} + 2\right\}, \tag{A.8}$$

where $\sigma = \sqrt{\frac{s-4m^2}{s}}$.
We again introduce the renormalization scale μ and find for $J(0)$, as $d \to 4$:

$$J(0) = -2\lambda - \frac{1}{16\pi^2}\left(\ln\frac{m^2}{\mu^2} + 1\right). \tag{A.9}$$

- To solve the integral

$$J_\mu(p) = -i\int d\tilde{q}\frac{q_\mu}{\left(m^2-q^2\right)\left(m^2-(p-q)^2\right)}, \tag{A.10}$$

change the integration variable from q to $q + \frac{p}{2}$. This yields

$$J_\mu(p) = -i\int d\tilde{q}\frac{q_\mu + \frac{p_\mu}{2}}{\left(m^2-(q+\frac{p}{2})^2\right)\left(m^2-(q-\frac{p}{2})^2\right)}.$$

The contribution with q_μ in the integrand vanishes for symmetry reasons. Undoing the change in the integration variable leads to

$$J_\mu(p) = \frac{p_\mu}{2}J(s). \tag{A.11}$$

This is quite a fast way to solve (A.10), though not the canonical one.

- Applying the Feynman-Schwinger trick, the integral

$$J_{\mu\nu}(p) = -i\int d\tilde{q}\frac{q_\mu\, q_\nu}{\left(m^2-q^2\right)\left(m^2-(p-q)^2\right)} \tag{A.12}$$

is brought to the form

$$-i\int_0^1 dx \int d\tilde{q}\frac{q_\mu q_\nu + x^2 p_\mu p_\nu}{\left(g^2(x,s)-q^2\right)^2}.$$

This yields two contributions. One is proportional to $p_\mu p_\nu$. Lorentz invariance implies that the other one is of the form $g_{\mu\nu}\cdot\beth$. Contracting with $g^{\mu\nu}$ yields

$$d\cdot\beth = -i\int_0^1 dx \int d\tilde{q}\frac{q^2}{\left(g^2(x,s)-q^2\right)^2}.$$

The integrals over q may be done applying equation (A.6). The final result for (A.12) can be expressed in terms of already known quantities:

$$J_{\mu\nu}(p) = -\frac{1}{d-1}\frac{p_\mu p_\nu}{s^2}\left(sI_1 - dI_2\right) + \frac{1}{d-1}\frac{g_{\mu\nu}}{s}\left(sI_1 - I_2\right) \tag{A.13}$$

where

$$I_1(s) = m^2 J(s) \quad \text{and} \quad I_2(s) = \frac{1}{4}\left(2si\Delta(0) + s^2 J(s)\right). \tag{A.14}$$

- The last two types of integrals occurring in the one-loop calculation of elastic $\pi\pi$-scattering are

$$-i\int d\tilde{q}\,\frac{q^2 q_\mu}{AB} = -i\int d\tilde{q}\,\frac{q_\mu(m^2-A)}{AB} = m^2 J_\mu(p) + i\int d\tilde{q}\,\frac{p_\mu+q_\mu}{m^2-k^2} = m^2 J_\mu(p) + p_\mu i\Delta(0) \quad \text{(A.15)}$$

and

$$-i\int d\tilde{q}\,\frac{q^2\cdot q^2}{AB} = -i\int d\tilde{q}\,\frac{q^2(A+m^2)}{AB} = m^2 J^\mu_\mu(p) - i\int d\tilde{q}\,\frac{q^2}{(p-q)^2-m^2} = m^2 J^\mu_\mu(p) - i\int d\tilde{q}\,\frac{(q+p)^2}{q^2-m^2} = m^2 J^\mu_\mu(p) - i\Box\Delta(0) + is\Delta(0), \quad \text{(A.16)}$$

where we have introduced the abbreviations $A = \left(m^2 - q^2\right)$ and $B = \left(m^2 - (p-q)^2\right)$ for simplicity.
In dimensional regularization $\delta(0)$ vanishes – equation (A.2) thus reads

$$\Box\Delta(0) = -m^2\Delta(0). \quad \text{(A.17)}$$

From equation (A.3), (A.4) and (A.9) we get

$$i\Delta(0) = m^2 J(0) + \frac{m^2}{16\pi^2}\ . \quad \text{(A.18)}$$

All one-loop integrals may therefore be expressed in terms of the function $J(s)$ and thus split up into a part proportional to $\bar{J}(s)$ and into another part proportional to $J(0)$.

- In the limit $d \to 4$ we get

$$\frac{J(0)}{d-1} = \frac{1}{3}\,J(0) + \frac{1}{72\pi^2} \qquad \text{and} \qquad \frac{d\,J(0)}{d-1} = \frac{4}{3}\,J(0) + \frac{1}{72\pi^2}\ . \quad \text{(A.19)}$$

- The corresponding expression for $J_{\mu\nu}(p) = \bar{J}_{\mu\nu}(p) + \tilde{J}_{\mu\nu}(p)$ is:

$$\bar{J}_{\mu\nu}(p) = \frac{\bar{J}(s)}{3}\left\{p_\mu p_\nu\left(1-\frac{m^2}{s}\right) + g_{\mu\nu}\left(m^2 - \frac{s}{4}\right)\right\}\ , \quad \text{(A.20)}$$

$$\tilde{J}_{\mu\nu}(p) = p_\mu p_\nu\left(\frac{J(0)}{3} + \frac{1}{288\pi^2}\right) + g_{\mu\nu}\left(J(0)\left(\frac{m^2}{2} - \frac{s}{12}\right) + \frac{m^2}{32\pi^2} - \frac{s}{288\pi^2}\right)\ .$$

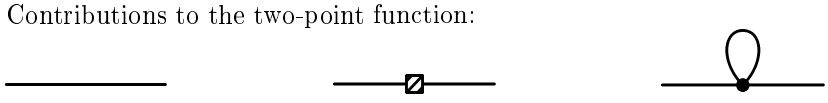
- When contracting the two indices we have to be careful to remember $g^\mu_\mu = d$ – therefore equation (A.12) has to be evaluated separately in this case. The result is

$$\bar{J}^\mu_\mu(p) = m^2\bar{J}(s)\,, \quad \text{and} \quad \tilde{J}^\mu_\mu(p) = 2m^2 J(0) + \frac{m^2}{16\pi^2}\ . \quad \text{(A.21)}$$

A.3 Feynman diagrams for elastic $\pi\pi$-scattering

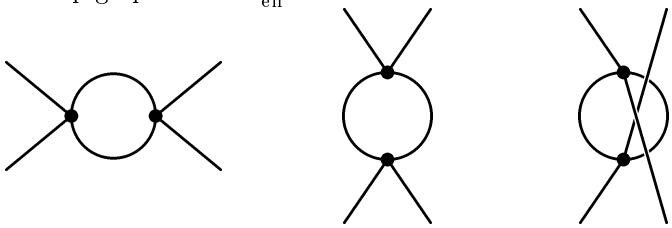
In the following all Feynman diagrams contributing to the elastic $\pi\pi$-scattering amplitude are shown. A full circle denotes a vertex from $\mathcal{L}_{\text{eff}}^{(2)}$, whereas $\mathcal{L}_{\text{eff}}^{(4)}$-vertices are marked by a shaded square.

- Contributions to the two-point function:



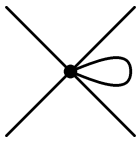
- Treelevel diagrams:

- One-loop graphs from $\mathcal{L}_{\text{eff}}^{(2)}$:



- External line insertions:

- Graph involving six pion fields:



A.4 Two mathematical notes

In this appendix we have a short look at two mathematical relations, widely used in deducing and solving dispersion relations.

- *Cauchy integral representation:*
  Let $f(z)$ be a function which is analytic in the whole complex plane, except for a cut starting at c running to infinity along the real axes. Assume that $f(z)$ is bounded by a power of $|z|$ as $|z| \to \infty$, i.e. that $f(z)/z^n$ tends to zero for large $|z|$, independently of the direction. According to Cauchy's theorem, the function $f(z)/P(z)$, where $P(z)$ denotes a polynomial of degree n with zeros z_i, allows the following integral representation:

$$\frac{f(z)}{P(z)} = \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{P(\xi)(\xi - z)} + \sum_i \frac{\mathrm{Res}\,(f/P, z_i)}{z - z_i}\,. \tag{A.22}$$

  The contour C runs above the real axes from c to $\infty + i\epsilon$, then around a circle going through $-\infty$ back to $\infty - i\epsilon$ and finally below the real axes from $\infty - i\epsilon$ to c. Provided that $f(z)/P(z)$ vanishes sufficiently fast for large $|z|$, the circle is harmless and we conclude that $f(z)$ can be written in the form

$$f(z) = Q(z) + \frac{P(z)}{\pi} \int_c^\infty \frac{\mathrm{disc}(f(x))\, dx}{P(x)(x - z)}\,. \tag{A.23}$$

  where the discontinuity of $f(x)$ across the cut is defined as

$$\mathrm{disc}(f(x)) = \frac{f(x + i\epsilon) - f(x - i\epsilon)}{2i}\,.$$

  The polynomial

$$Q(z) = P(z) \sum_i \frac{\mathrm{Res}(f(x)/P(x), z_i)}{z - z_i}$$

  is of order $n - 1$ and is called a subtraction polynomial. Accordingly the relation (A.23) is said to have n subtractions.

- *Sokhotsky-Plemelj formula:*
  This is a standard distribution relation.

$$\frac{1}{x \pm i\epsilon} = \mp i\delta(x) + \mathcal{P}\frac{1}{x}\,. \tag{A.24}$$

  This is readily proven by deforming the integration path by replacing the straight line between $-\delta$ and δ with a semicircle lying entirely in the opposite half-plane than the pole and of radius δ and performing the limit $\delta \to 0$.

References

[1] A.V. Anisovich, H. Leutwyler. *Dispersive analysis of the decay $\eta \to 3\pi$.* Physics Letters B 375 (1996) 335.

[2] A.V. Anisovich, H. Leutwyler. *Measuring the quark mass ratio $\frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$ by means of $\eta \to 3\pi$.* Unpublished draft.

[3] Steven Weinberg. *The Quantum Theory of Fields, Volume I.* Cambridge University Press 1996.

[4] Steven Weinberg. *The Quantum Theory of Fields, Volume II.* Cambridge University Press 1996.

[5] Lewis H. Ryder. *Quantum Field Theory.* Cambridge University Press 1996.

[6] J. Gasser and H. Leutwyler. *Chiral perturbation theory: expansions in the mass of the strange quark.* Nuclear Physics B250 (1985) 465.

[7] J. Gasser and H. Leutwyler. *$\eta \to 3\pi$ to one loop.* Nuclear Physics B250 (1985) 539.

[8] R. Baur, J. Kambor and D. Wyler. *Electromagnetic Corrections to the decays $\eta \to 3\pi$.* ZU-TH 16/95, IPNO/TH 95-53, hep-ph/9510396.

[9] H. Leutwyler. *Goldstone Bosons.* BUTP-94/17, hep-ph/9409422.

[10] H. Leutwyler. *Foundations and scope of chiral perturbation theory.* BUTP-94/18, hep-ph/9409423.

[11] H. Leutwyler. *Chiral Effective Lagrangians.* BUTP-91/26.

[12] H. Leutwyler. *Principles of Chiral Perturbation Theories.* BUTP-94/13.

[13] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M.E. Sainio. *Pion-pion scattering at low energy.* BUTP-97/17, hep-ph/9707291.

[14] G. Ecker. *Effective Field Theories and Chiral Perturbation Theory.* Lectures given in course of *"Troisième Cycle de la Physique en Suisse Romande".*

[15] Gabriel Barton. *Introduction to Dispersion Techniques in Field Theory.* W. A. Benjamin, Inc.; New York, Amsterdam, 1965.

[16] *Review of Particle Physics.* The European Physical Journal C. Volume 3, Number 1-4, 1998

[17] J. Stern, H. Sazdjian and N.H. Fuchs *Phys. Rev. D 47 (1993) 3814.*